Holographic stress tensor for non-relativistic theories

JHEP09(2009)009

(http://iopscience.iop.org/1126-6708/2009/09/009)

Published by IOP Publishing for SISSA

Received: *July 27, 2009*
Accepted: *August 21, 2009*
Published: *September 1, 2009*

Holographic stress tensor for non-relativistic theories

Simon F. Ross[a] **and Omid Saremi**[b]

[a] *Centre for Particle Theory, Department of Mathematical Sciences, Durham University, South Road, Durham DH1 3LE, U.K.*

[b] *Department of Physics, McGill University, 3600 University St, Montréal, Que H3A 2T8, Canada*

E-mail: S.F.Ross@durham.ac.uk, omid@hep.physics.mcgill.ca

Abstract: We discuss the calculation of the field theory stress tensor from the dual geometry for two recent proposals for gravity duals of non-relativistic conformal field theories. The first of these has a Schrödinger symmetry including Galilean boosts, while the second has just an anisotropic scale invariance (the Lifshitz case). For the Lifshitz case, we construct an appropriate action principle. We propose a definition of the non-relativistic stress tensor complex for the field theory as an appropriate variation of the action in both cases. In the Schrödinger case, we show that this gives physically reasonable results for a simple black hole solution and agrees with an earlier proposal to determine the stress tensor from the familiar AdS prescription. In the Lifshitz case, we solve the linearised equations of motion for a general perturbation around the background, showing that our stress tensor is finite on-shell.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence

ArXiv ePrint: 0907.1846

doi:10.1088/1126-6708/2009/09/009

Contents

1 Introduction

The use of gravitational duals to study strongly-coupled field theories [1, 2] has produced substantial progress in our understanding of both vacuum correlation functions and finite-temperature behaviour at strong coupling. The domain in which this holographic toolbox has been put into use is remarkably large. For instance, the hydrodynamic limit of the duality has proved insightful in studying the quark-gluon plasma created at RHIC [3–5]. There have also been attempts to model interesting condensed matter systems using a corresponding gravitational dual [6–9]. Much of this work has concerned relativistic theories with a conformal symmetry in the ultraviolet, which are described by asymptotically Anti-de Sitter (AdS) spacetimes. Largely inspired by condensed matter systems, however, this has recently been extended to consider non-relativistic theories with an anisotropic scaling symmetry

$$t \to \lambda^z t, \quad x^i \to \lambda x^i. \tag{1.1}$$

The case $z = 2$, which is the symmetry of a free non-relativistic theory (as the Hamiltonian is quadratic in the momenta), is often of particular interest. Strongly-coupled theories with this scaling symmetry arise as critical points in condensed matter systems: two cases of interest are where the theory has a Galilean boost symmetry together with the anisotropic scaling symmetry, forming the Schrödinger symmetry group for $z = 2$ [10–12], and where the theory has no boost symmetry, which we will refer to as the Lifshitz case [13–15]. Dual geometries realising these symmetries as isometries were obtained for the Schrödinger symmetry in [16, 17] and for the Lifshitz case in [18].

These geometries are, of course, not asymptotically AdS. This provides an additional motivation for studying these systems, as the generalisation of holographic techniques to this new context may offer new insights into the nature of the relation between quantum gravity in asymptotically non-AdS spacetimes and the dual field theory. It also requires the development of a new dictionary relating bulk to the boundary quantities.

In the asymptotically AdS case, there is a well-developed framework for calculating field theory quantities from the bulk spacetime. The key element in this framework is an appropriate action principle for the bulk theory, which is finite on-shell and stationary under variations which satisfy some asymptotic fall-off conditions. This is constructed by adding covariant local boundary counter-terms to the bulk action [19, 20]. Correlation functions for the field theory can then be obtained by considering appropriate variations of this action with respect to the boundary data. One important example is the expectation value of the stress tensor, which plays a central role in the application to finite-temperature field theory in particular. The stress tensor is obtained by variation of the action with respect to the boundary metric [19, 20]. This prescription has been extensively used in the context of AdS/CFT and elsewhere. More recently, in one particularly interesting application, it was applied to obtain a very beautiful and direct relationship between the dynamics of the stress tensor in the hydrodynamic regime in the field theory and the equations of motion of the bulk gravitational theory [21, 22].

Some progress has been made in extending these aspects of the holographic dictionary to the asymptotically Schrödinger case. A black hole solution corresponding to the finite temperature grand canonical ensemble in the field theory was constructed in [23–25]. An action principle for asymptotically Schrödinger spaces was constructed in [25], by adding local covariant boundary counter-terms to the bulk action as in the AdS case. However, a stress tensor was not successfully constructed from the variation of this action. The asymptotically Schrödinger solutions are obtained by applying a solution-generating transformation to asymptotically AdS solutions, and it was proposed in [23] that the stress tensor obtained from the asymptotically AdS solution could be re-interpreted in terms of the non-relativistic solution. This approach was used to study the hydrodynamic regime in this theory in [26] by re-using the results of [21]. For the Lifshitz case, black hole solutions were obtained in [27, 28], and the energy of these solutions was studied in [29], but an action principle and stress tensor have not yet been obtained for this theory.

To find a detailed map between bulk fluctuations and field theory objects, one might need to find embeddings of these spacetimes in a complete theory of quantum gravity like string theory. This was accomplished for the Schrödinger case in [23–25]. We will not attempt to do this for the Lifshitz case here. Rather we study simply the generalisation of the holographic dictionary at the level of the classical gravity in the bulk.

The aim of this paper is to further develop the holographic dictionary for asymptotically Schrödinger and asymptotically Lifshitz spacetimes, focusing on the construction of one-point functions. We only consider the case where the boundary metric is flat; the extension to consider more general boundary metrics, and in particular the case where the boundary metric is a sphere, is an interesting problem for the future. We will construct an appropriate action principle for the Lifshitz case in section 2. We then propose a definition of the non-relativistic stress tensor complex for the field theory which can be applied to both Lifshitz and Schrödinger cases. A key element of our definition is treating the variation of the matter fields appropriately. Our approach is strongly inspired by [30], which showed that in the relativistic case in the presence of arbitrary bulk matter fields, the stress tensor is defined by considering the variation of the boundary metric holding fixed the tangent space components of the matter fields. We propose to apply the same prescription to the non-relativistic cases. Considering the variation of the boundary geometry with the tangent space components of the matter fields fixed turns out to be crucial to obtain a finite stress tensor. We discuss the application of this prescription to calculate the stress tensor in the Lifshitz case in general in section 3, and apply these ideas to asymptotically Schrödinger spacetimes in section 4. In the Schrödinger case, we show that the results obtained from our proposal agree with those obtained from the stress tensor of the asymptotically AdS solution following the prescription of [23].

In section 5, we solve the bulk equations of motion for a general linearised perturbation of the Lifshitz spacetime, and calculate our stress tensor for this linearised perturbation. We find that the stress tensor for the linearised perturbations is finite. The finiteness of the stress tensor is an important test of our prescription. We solve the bulk equations of motion for the perturbation in a series expansion in derivatives of the perturbation along the boundary directions. In the linearised analysis, only a finite number of orders in this expansion make finite contributions to the boundary stress tensor.[1] If we considered a general perturbation, the departure from the background solution would be small in the asymptotic regime, so for perturbations that fall off sufficiently rapidly at large distances, this linearised analysis gives a relation between the asymptotic behaviour of the perturbation in the bulk and the stress tensor in the dual field theory, analogous to that given by the Fefferman-Graham expansion in the asymptotically AdS case. Note however that for $z \geq 2$, the falloff of some parts of the bulk perturbation is too slow for this linearised analysis to be justified, and a full non-linear analysis will be required even just to relate the asymptotic falloff of the fields to the boundary stress tensor. This also occurs for the asymptotically Schrödinger case.

We conclude with a summary of our results and a discussion of issues and directions for further development in section 6. In appendix A, we calculate the contribution to the stress tensor for asymptotically Lifshitz spacetimes from counterterms involving derivatives of the boundary fields. In appendix B, as yet another consistency check, we show that our definition of the energy density following from the stress tensor complex agrees with

[1] This is different from the hydrodynamic analysis, where all orders in derivatives contribute, because we are linearising around the zero-temperature Lifshitz geometry of [18], not around a black hole solution.

the thermodynamic energy which would be obtained from the Euclidean action for static asymptotically Lifshitz black holes.

2 Action for Lifshitz case

In [18], it was proposed that a holographic dual to a theory with the anisotropic scaling symmetry (1.1) and no boost symmetry could be obtained by considering the metric[2]

$$ds^2 = -r^{2z}dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{r^2}, \tag{2.1}$$

where the scaling symmetry is realised as an isometry: $t \to \lambda^z t, x^i \to \lambda x^i, r \to \lambda^{-1} r$. This was realised in [18] as a solution of a theory with two p-form gauge fields, with a Chern-Simons coupling between the two gauge fields. In [31], it was observed that one could construct a simpler theory with the metric and a massive vector field by integrating out one of the p-form gauge fields. We will consider this case, as it usefully restricts the form of the counter-terms we can consider in constructing an action principle. The equations of motion for this theory are

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + \frac{1}{2}F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{8}F_{\lambda\rho}F^{\lambda\rho}g_{\mu\nu} + \frac{1}{2}m^2 A_\mu A_\nu \tag{2.2}$$

and

$$\nabla_\mu F^{\mu\nu} = m^2 A^\nu. \tag{2.3}$$

If we choose $\Lambda = -\frac{1}{2}(z^2 + z + 4)$ and $m^2 = 2z$, this theory has a solution

$$ds^2 = -r^{2z}dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{r^2}, \quad A = \alpha r^z dt, \quad \alpha^2 = \frac{2(z-1)}{z}. \tag{2.4}$$

It is straightforward to extend the analysis to a general number of spatial dimensions, but we will focus on the case of two spatial dimensions for simplicity. We keep z general; in the linearised analysis we will find that $z = 2$ is a special case, where some aspects of the analysis need separate treatment.

We want to define an action for this theory which satisfies $\delta S = 0$ with appropriate boundary conditions by adding appropriate local counter-terms. To preserve the diffeomorphism invariance of the action, these counter-terms should be covariant in the boundary fields. We consider

$$\begin{aligned} S = S_{\text{bulk}} + S_{\text{bdy}} = & \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right) \\ & + \frac{1}{16\pi G_4}\int d^3\xi\sqrt{-h}(2K - 4 + f(A_\alpha A^\alpha)) + S_{\text{deriv}}, \end{aligned} \tag{2.5}$$

where ξ^α are coordinates on the boundary at some constant r, $h_{\alpha\beta}$ is the induced metric, and $K_{\alpha\beta} = \nabla_{(\alpha}n_{\beta)}$ is the extrinsic curvature of the boundary, where the unit vector n^μ is orthogonal to the boundary and outward-directed. S_{deriv} is a collection of terms

[2] We always use coordinates such that the boundary is at $r = \infty$.

involving derivatives of the boundary fields, which could involve both the curvature tensor constructed from the boundary metric and covariant derivatives of A_α. Since the boundary fields are constants for (2.4), as the boundary is flat, this part of the action will not contribute to the on-shell value of the action for the pure Lifshitz solution or its first variation around the Lifshitz background. We can therefore ignore it for this section, but it can play a role when we come to consider general asymptotically Lifshitz spacetimes. The only scalar we can build from A on the boundary is $A_\alpha A^\alpha$, as A is constant along the boundary. For the Lifshitz spacetime, $A_\alpha A^\alpha = -\alpha^2$ is constant, so any function of this scalar will contribute to the action at the same order in r at large r, which is why we consider an arbitrary function $f(A^\alpha A_\alpha)$ in our boundary term. For simplicity, we will choose units such that $16\pi G_4 = 1$ henceforth.

The variation of the action about a solution of the equations of motion is just a boundary term,

$$\delta S = \int d^3\xi\sqrt{-h}\Big[(\pi_{\alpha\beta} + 2h_{\alpha\beta})\delta h^{\alpha\beta} - n^\mu F_{\mu\nu}\delta A^\nu \\ + f'(A_\alpha A^\alpha)(2A_\alpha \delta A^\alpha + A_\alpha A_\beta \delta h^{\alpha\beta}) - \frac{1}{2}f(A_\alpha A^\alpha)h_{\alpha\beta}\delta h^{\alpha\beta}\Big], \tag{2.6}$$

where $\pi_{\alpha\beta} = K_{\alpha\beta} - Kh_{\alpha\beta}$. For the Lifshitz spacetime (2.4), $\pi_{tt} + 2h_{tt} = 0$, $\pi_{ij} + 2h_{ij} = (1-z)r^2\delta_{ij}$, and $n^\mu F_{\mu\nu}\delta A^\nu = z\alpha r^z \delta A^t$. Therefore, there are variations involving δh^{ij} and δA^t that we need to cancel. However, the variation involving δh^{tt} has already canceled. To avoid generating a new one from the terms involving $f(A_\alpha A^\alpha)$, we must have $f(A_\alpha A^\alpha) = \beta\sqrt{-A_\alpha A^\alpha}$ (so that the $\sqrt{h^{tt}}$ in this cancels the $\sqrt{h_{tt}}$ in the overall $\sqrt{-h}$ to give us a term which does not involve h_{tt}). Requiring the cancellation of the other terms determines $\beta = -z\alpha$. The action is thus

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right) \\ + \int d^3\xi\sqrt{-h}(2K - 4 - z\alpha\sqrt{-A_\alpha A^\alpha}) + S_{\text{deriv}}. \tag{2.7}$$

It is remarkable that fixing a single coefficient suffices to cancel both the divergences associated with δh^{ij} and δA^t. Let us define

$$s_{\alpha\beta} = \sqrt{-h}\left[(\pi_{\alpha\beta} + 2h_{\alpha\beta}) + \frac{z\alpha}{2}(-A_\alpha A^\alpha)^{-1/2}(A_\alpha A_\beta - A_\gamma A^\gamma h_{\alpha\beta})\right] + s^{\text{deriv}}_{\alpha\beta}, \tag{2.8}$$

$$s_\alpha = -\sqrt{-h}(n^\mu F_{\mu\alpha} - z\alpha(-A_\alpha A^\alpha)^{-1/2}A_\alpha) + s^{\text{deriv}}_\alpha. \tag{2.9}$$

Then the general variation of the action is

$$\delta S = \int d^3\xi(s_{\alpha\beta}\delta h^{\alpha\beta} + s_\alpha \delta A^\alpha). \tag{2.10}$$

In the background (2.4), we have $s_{\alpha\beta} = 0$, $s_\alpha = 0$ due to cancellations between the different terms, and this action satisfies $\delta S = 0$ for arbitrary variations around (2.4). It also has $S = 0$ for (2.4).

Thus, we have a finite on-shell action which defines a good variational principle for our background spacetime. Note that an asset of working with the massive vector theory is that the form of the possible local counterterms is tightly constrained; with the original theory of [18], we could build several different scalars from the two p-forms, and it would not be so obvious what a convenient form for the action is. In section 5, we will show that the action is finite on-shell and gives a well-defined variational principle for a class of asymptotically Lifshitz spacetimes. Before doing so, however, we want to address the calculation of the stress tensor from the action in asymptotically Lifshitz and asymptotically Schrödinger spacetimes.

3 Stress tensor

A core element of the holographic renormalization programme in the gauge-gravity duality for relativistic field theories is that given a well-defined action principle, we can use it to define a boundary stress tensor as the variation of the action with respect to the boundary metric [19, 20]. The resulting stress tensor has been shown to define conserved charges which generate the asymptotic symmetries of the geometry in very general circumstances [30]. This stress tensor carries important physical information about the dual field theory. In this section, we want to discuss the calculation of such a boundary stress tensor from a bulk action principle in the non-relativistic case. We will focus explicitly on the Lifshitz example in this section, as its treatment is simpler, but similar ideas apply to asymptotically Schrödinger spacetimes, which we consider in the next section.

For asymptotically Lifshitz spacetimes, the dual field theory is non-relativistic, so it will not have a covariant relativistic stress tensor, but we would still expect it to have a stress tensor complex, consisting of the energy density $\mathcal{E}$, energy flux $\mathcal{E}_i$, momentum density $\mathcal{P}_i$ and spatial stress tensor Π_{ij}, satisfying the conservation equations

$$\partial_t \mathcal{E} + \partial_i \mathcal{E}^i = 0, \quad \partial_t \mathcal{P}_j + \partial_i {\Pi^i}_j = 0. \tag{3.1}$$

We would like to derive such a stress tensor complex by considering some appropriate variations of the action principle we introduced in the previous section. Since the boundary theory is non-relativistic, the boundary data does not include a non-degenerate metric; the nonuniform r-dependence of the metric in the bulk along the boundary directions leads to a degenerate boundary metric. It is therefore not a priori obvious how to define the stress tensor complex. In this section we will follow the relativistic analysis as closely as possible; we postpone discussion of the appropriateness of this approach from the boundary theory point of view to the conclusions.

Since the background (2.4) involves a vector field, we will need to consider how this effects the definition of the stress tensor. This issue was considered in the relativistic case in [30], where it was argued that the appropriate definition of the stress tensor in the presence of tensor fields was to consider the variation of a boundary frame field $\hat{e}^{(A)}_\alpha$, holding the tangent space components $\phi^{[i]}_{AB\ldots}$ of the other fields fixed where A, B,$\cdots$ denote tangent space directions and i denotes matter species. This was shown to provide a stress tensor whose integrals give the conserved charges generating asymptotic symmetries and which is conserved up to terms involving derivatives of the other fields [30].

To be more specific, if we considered a background with a massive vector field which was dual to a relativistic field theory, we should hold the components A_A of the vector with tangent space indices fixed. We would then write the general variation of the action as

$$\delta S = \int \hat{\epsilon}(T^{\alpha}_{A}\delta\hat{e}^{(A)}_{\alpha} + s_A \delta A^A), \tag{3.2}$$

where $\hat{e}^{(A)}_{\alpha}$ is a boundary frame field defining the boundary metric, and $\hat{\epsilon}$ is the associated volume form on the boundary. That is, $\hat{e}^{(A)}_{\alpha}$ are the components of the frame along the boundary directions, rescaled by an appropriate power of r such that $\hat{e}^{(A)}_{\alpha}$ have finite limits as $r \to \infty$. In an asymptotically AdS spacetime, the choice of $\hat{e}^{(A)}_{\alpha}$ corresponds to the choice of the boundary metric $g_{(0)}$ appearing in the expansion of the asymptotic geometry in Fefferman-Graham coordinates,

$$ds^2_{\mathrm{AdS}} = \frac{dr^2}{r^2} + r^2[g_{(0)\alpha\beta} + \mathcal{O}(r^{-2})]dx^\alpha dx^\beta, \tag{3.3}$$

and the bulk frame fields are related to the boundary frame fields by $e^{(A)} = r\hat{e}^{(A)}$, $e^{(r)} = \frac{dr}{r}$. The stress tensor T^{α}_{A} was shown in [30] to be conserved up to terms involving the variation of the matter fields,

$$D_\alpha T^{\alpha}_{\beta} = s_A \partial_\beta A^A, \tag{3.4}$$

where D_α is the covariant derivative on the boundary defined by requiring $D_\alpha \hat{e}^{(B)}_{\beta} = 0$. In the asymptotically AdS case, the key advantage of the prescription of [30] is that it gives a stress tensor which is conserved in this sense. If we considered the stress tensor as defined by considering the variation of the metric holding the spacetime components of the matter fields fixed, we would obtain a finite result, but there would be additional terms on the right-hand side of this conservation equation, and as a result, the stress tensor would not in general give rise to the correct conserved charges (although the difference is unimportant in many common examples). In the non-relativistic cases, as we will see below, this distinction is much more important, and we must follow the prescription of [30] to obtain finite results for the stress tensor complex.

We want to apply a similar prescription to our non-relativistic cases. In asymptotically Lifshitz spacetimes, because of the different scaling of the time and space directions, there is no non-degenerate boundary metric that we can associate with the boundary at $r = \infty$ in our spacetime. However, when we calculate the variation in (2.7), we first cut off the spacetime at some finite radius r, and then consider the limit as $r \to \infty$. At finite r, there is a well-defined boundary metric. We could rescale the bulk metric by r^2 so that the spatial parts have a well-defined large r limit; the additional factor of $r^{2(z-1)}$ multiplying dt^2 can then be thought of as a radius-dependent speed of light, so that the limit $r \to \infty$ corresponds to taking the speed of light to infinity in the boundary theory. In the non-relativistic limit of a relativistic field theory, we can recover both of the conservation equations (3.1) from the conservation of the relativistic stress tensor. If we take this point of view, then we should expect to be able to define the non-relativistic stress tensor complex following essentially the same recipe as in the relativistic case.[3]

For the Lifshitz case, we assume that we have a bulk orthonormal frame with components

$$e^{(0)} = r^z \hat{e}^{(0)}, \quad e^{(i)} = r\hat{e}^{(i)}, \quad e^{(3)} = \frac{dr}{r}, \tag{3.5}$$

and a massive vector A_M. From the heuristic point of view above, the different scaling in $e^{(0)}$ compared to $e^{(i)}$ corresponds to a scaling by the radius-dependent speed of light on the surface at constant r. We will use indices $M = 0, 1, 2, 3$ to denote frame components and $\mu = t, x, y, r$ to denote spacetime components. The spacetime will asymptotically approach the pure Lifshitz solution (2.4) if $\hat{e}^{(0)} \to dt$, $\hat{e}^{(i)} \to dx^i$ and $A^M \to \alpha\delta_0^M$ as $r \to \infty$.[4]

We therefore construct the stress tensor complex for the non-relativistic theories by regarding $\hat{e}^{(0)}$, $\hat{e}^{(i)}$ and A^M (more accurately, their limits as $r \to \infty$) as the boundary data, and defining

$$\delta S = \int \hat{\epsilon}[-\mathcal{E}\delta\hat{e}_t^{(0)} - \mathcal{E}^i\delta\hat{e}_i^{(0)} + \mathcal{P}_i\delta\hat{e}_t^{(i)} + \Pi_i^j\delta\hat{e}_j^{(i)} + s_A\delta A^A]. \tag{3.6}$$

As in the relativistic case, we expect that the energy density, energy flux, momentum density and spatial stress tensor so defined will satisfy the conservation equations (3.1) up to terms involving the variation of the massive vector field. The treatment of the matter fields, holding the components with tangent space indices fixed, turns out to be crucial to obtain finite results for the stress tensor.

If the boundary data are taken to be $\hat{e}^{(0)} \to dt$, $\hat{e}^{(i)} \to dx^i$, then $\hat{\epsilon}$ is just the flat volume form $d^3\xi$, and we can rewrite the above definitions in terms of the coefficients $s_{\alpha\beta}$ and s_α that we used to write the general metric variation in (2.7):

$$\mathcal{E} = 2s^t{}_t - s^t A_t, \quad \mathcal{E}^i = 2s^i{}_t - s^i A_t, \tag{3.7}$$

and

$$\mathcal{P}_i = -2s^t{}_i + s^t A_i \quad \Pi_i^j = -2s^j{}_i + s^j A_i, \tag{3.8}$$

where we have multiplied through by factors of the frame fields to simplify the form of these expressions, taking advantage of the fact that the frame fields each have a single component to leading order in the large r limit, so all indices are now spacetime indices. Note that when $z = 1$, $\alpha = 0$ and these definitions reduce to the familiar AdS rules.

Normally, to obtain finite quantities in the non-relativistic limit of the relativistic stress tensor, we need to eliminate divergent contributions to the energy density and energy flux coming from the rest mass of the particles (see e.g. [33] chapter 15). However, these Lifshitz theories do not have Galilean boost invariance, and hence do not conserve particle number. We will find below that the above definitions give a finite result for the energy density, indicating that there is no divergent contribution from rest mass that we need to eliminate.

[3]Working on a finite cutoff surface in this way is also similar in spirit to the analysis of holographic renormalisation for asymptotically flat spaces in [32].

[4]This is a necessary condition; we will give a more precise definition of asymptotically Lifshitz boundary conditions for more general boundary data later.

4 Schrödinger spacetimes

Another example of non-relativistic holography is the Schrödinger spacetime [16, 17],

$$ds^2 = -r^4(dx^+)^2 + r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2}. \tag{4.1}$$

This solution has the Schrödinger symmetry group as its isometries (including in particular the anisotropic scaling symmetry (1.1) with $z = 2$, when we identify t there with x^+). It was shown in [34, 35] that this symmetry group essentially uniquely determines this form for the metric.

A simple action which has a solution with this metric is [25]

$$\begin{aligned} S &= \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R - \frac{4}{3}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4A_\mu A^\mu - V(\phi)\right] \\ &\quad + \frac{1}{16\pi G_5}\int d^4\xi\sqrt{-h}\left[2K - 6 + (1+c_4\phi)A_\mu A^\mu + c_5(A_\mu A^\mu)^2 + (2c_4 - 4c_5 + 3)\phi^2\right], \end{aligned} \tag{4.2}$$

which gives

$$\delta S = \frac{1}{16\pi G_5}\int d^4\xi(s_{\alpha\beta}\delta h^{\alpha\beta} + s_\alpha \delta A^\alpha + s_\phi\delta\phi), \tag{4.3}$$

with

$$\begin{aligned} s_{\alpha\beta} &= \sqrt{-h}\Big[\pi_{\alpha\beta} + 3h_{\alpha\beta} + (1+c_4\phi)\Big(A_\alpha A_\beta - \frac{1}{2}A_\gamma A^\gamma h_{\alpha\beta}\Big) \\ &\qquad + c_5 A_\delta A^\delta\Big(2A_\alpha A_\beta - \frac{1}{2}A_\gamma A^\gamma h_{\alpha\beta}\Big) - \frac{1}{2}(2c_4 - 4c_5 + 3)\phi^2 h_{\alpha\beta}\Big], \end{aligned} \tag{4.4}$$

$$s_\alpha = \sqrt{-h}(-n^\mu F_{\mu\alpha}e^{-8\phi/3} + 2(1 + c_4\phi + 2c_5 A_\gamma A^\gamma)A_\alpha), \tag{4.5}$$

and

$$s_\phi = \sqrt{-h}\Big(-\frac{8}{3}n^\mu\partial_\mu\phi + c_4 A_\alpha A^\alpha + 2(2c_4 - 4c_5 + 3)\phi\Big). \tag{4.6}$$

This has an asymptotically Schrödinger black hole solution [23–25]. The metric is

$$\begin{aligned} ds_E^2 &= r^2\, k(r)^{-\frac{2}{3}}\left(\left[\frac{1-f(r)}{4\beta^2} - r^2\, f(r)\right](dx^+)^2 + \frac{\beta^2 r_+^4}{r^4}(dx^-)^2 - [1+f(r)]\, dx^+\, dx^-\right) \\ &\quad + k(r)^{\frac{1}{3}}\left(r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2\, f(r)}\right), \end{aligned} \tag{4.7}$$

with the massive vector and scalar

$$\begin{aligned} A &= \frac{r^2}{k(r)}\left(\frac{1+f(r)}{2}\,dx^+ - \frac{\beta^2 r_+^4}{r^4}\,dx^-\right), \\ e^\phi &= \frac{1}{\sqrt{k(r)}}\,. \end{aligned} \tag{4.8}$$

This solution is obtained by applying a solution-generating transformation (the TsT transformation) to an asymptotically AdS vacuum black hole solution. In [25], it was shown that the action (4.2) is finite and satisfies $\delta S = 0$ for this black hole solution. However, some of the coefficients $s_{\alpha\beta}$ in the variation diverge, so a naive attempt to define the stress tensor will fail [25, 36].

As in the asymptotically Lifshitz case, there is no non-degenerate boundary metric for the asymptotically Schrödinger spacetimes. However, as before, there is a non-degenerate metric on the surfaces of finite r, which degenerates in the limit as $r \to \infty$. We therefore define a stress tensor complex for these spacetimes by adapting the relativistic prescription in [30]. In this case, the non-relativistic theory is meant to be obtained from the relativistic theory by light-cone reduction, with the momentum along the light cone direction interpreted as the conserved mass density ρ, which satisfies a conservation equation involving the mass flux ρ^i. The combination which appears as the coefficient of $\delta e_\alpha^{(A)}$ in δS is again $-2s^\alpha{}_A + s^\alpha A_A = (-2s^\alpha{}_\beta + s^\alpha A_\beta) e^\beta{}_{(A)}$. There is no obvious convenient choice of orthonormal frame. We therefore identify the components of the stress tensor complex in this case as

$$\mathcal{E} = 2s^+{}_+ - s^+ A_+, \qquad \mathcal{E}^i = 2s^i{}_+ - s^i A_+, \tag{4.9}$$
$$\mathcal{P}_j = -2s^+{}_j + s^+ A_j, \qquad \Pi^i{}_j = -2s^i{}_j + s^i A_j, \tag{4.10}$$

and

$$\rho = -2s^+{}_- + s^+ A_-, \quad \rho^i = -2s^i{}_- + s^i A_-, \tag{4.11}$$

where all the indices are again spacetime indices, and we have set $16\pi G_5 = 1$.

For the black hole solution (4.7), all of the vector components of the stress tensor complex vanish, and we find

$$\mathcal{E} = r_+^4, \quad \Pi_{xx} = \Pi_{yy} = r_+^4, \quad \rho = 2\beta^2 r_+^4, \tag{4.12}$$

in agreement with previous results obtained by different methods [25, 26]. Note that because of the slow falloff relative to the background, there is a potential finite $\beta^4 r_+^8$ term in $\mathcal{E}$, that is, a piece which comes from terms quadratic in the departure from the background. It is a non-trivial test of our definition of the stress tensor that this term vanishes.

For these asymptotically Schrödinger spacetimes, it was proposed in [23] that the stress tensor could be obtained by taking the stress tensor for the corresponding asymptotically AdS spacetime and taking the light cone reduction of it. This idea was applied to the study of the hydrodynamics for the non-relativistic theories with Schrödinger symmetry in [26]. It is therefore important for us to compare this approach to our new proposal.

These two approaches a priori look quite different; one reason why we might nevertheless expect agreement is that the stress tensor was shown in [30] to give the conserved charges associated with the asymptotic symmetries of the spacetime. In the Schrödinger case, the action of symmetries like time translation will commute with the TsT transformation, so we can perform a time translation by transforming to the asymptotically AdS spacetime, performing a time translation there, and transforming back to the asymptotically Schrödinger spacetime. Thus, the conserved charge obtained from the stress tensor of [23], which generates time translation in the asymptotically AdS spacetime, is naturally identified with the conserved charge which generates time translation in the asymptotically

Schrödinger spacetime. This provides some physical motivation for agreement of the two stress tensors.

For simplicity, let us consider an asymptotically Schrödinger spacetime which is obtained by a TsT transformation from a vacuum asymptotically AdS spacetime. This does not give the most general asymptotically Schrödinger spacetime (which would require us to consider an asymptotically AdS spacetime with non-zero scalar and massive vector fields in the bulk), but restricting consideration to this case leads to much simpler expressions, and includes all the examples that have been explicitly considered so far in the literature. If we start with a vacuum asymptotically AdS solution with metric

$$ds^2_{AdS} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta + \frac{dr^2}{r^2}, \tag{4.13}$$

and we assume that the metric is independent of a coordinate x^- which becomes null at large distances, then by applying a TsT transformation we will obtain an asymptotically Schrödinger solution with scalar field

$$e^{-2\phi} = 1 + \bar{g}_{--}, \tag{4.14}$$

massive vector field

$$A_\mu = e^{2\phi} \bar{g}_{\mu -}, \tag{4.15}$$

and metric

$$g_{\mu\nu} = e^{-2\phi/3} \bar{g}_{\mu\nu} - e^{4\phi/3} \bar{g}_{\mu -} \bar{g}_{\nu -}, \tag{4.16}$$

which implies the inverse metric is

$$g^{\mu\nu} = e^{2\phi/3} (\bar{g}^{\mu\nu} + \delta^\mu_- \delta^\nu_-). \tag{4.17}$$

Our definition of the non-relativistic stress tensor complex for the asymptotically Schrödinger spacetime corresponds to considering the light cone reduction of a "stress tensor"

$$T^\alpha_{\ \beta} = s^\alpha_{\ \beta} - \frac{1}{2} s^\alpha A_\beta = \sqrt{-h} h^{\alpha\gamma} \tau_{\gamma\beta} \tag{4.18}$$

where

$$\begin{aligned} \tau_{\gamma\beta} = \; & \pi_{\gamma\beta} + \frac{1}{2} e^{-8\phi/3} n^\mu F_{\mu\gamma} A_\beta \\ & - \frac{1}{2} \left(-6 + (1 + c_4 \phi) A_\delta A^\delta + c_5 (A_\delta A^\delta)^2 + (2c_4 - 4c_5 + 3)\phi^2 \right) h_{\gamma\beta}, \end{aligned} \tag{4.19}$$

whereas [23] would consider the stress tensor for the asymptotically AdS spacetime, which is simply

$$\bar{T}^\alpha_{\ \beta} = \sqrt{-\bar{h}} \bar{h}^{\alpha\gamma} [\bar{\pi}_{\gamma\beta} + 3\bar{h}_{\gamma\beta}]. \tag{4.20}$$

To compare these two, let's rewrite our stress tensor using the expression for the Schrödinger fields in terms of the AdS metric. The unit normal in the asymptotically Schrödinger geometry is $n^\mu = r e^{\phi/3} \delta^\mu_r$, so

$$\begin{aligned} K_{\alpha\beta} = \; & \frac{1}{2} r e^{\phi/3} \Big(-\frac{2}{3} \partial_r \phi e^{-2\phi/3} \bar{h}_{\alpha\beta} + e^{-2\phi/3} \partial_r \bar{h}_{\alpha\beta} \\ & - \frac{4}{3} \partial_r \phi e^{4\phi/3} \bar{h}_{\alpha -} \bar{h}_{\beta -} - e^{4\phi/3} \partial_r \bar{h}_{\alpha -} \bar{h}_{\beta -} - e^{4\phi/3} \bar{h}_{\alpha -} \partial_r \bar{h}_{\beta -} \Big), \end{aligned} \tag{4.21}$$

which gives

$$\pi_{\alpha\beta} = e^{-\phi/3}\bar{\pi}_{\alpha\beta} + \frac{1}{2} r e^{5\phi/3}\left(-2\partial_r\phi\bar{h}_{\alpha-}\bar{h}_{\beta-} - \partial_r\bar{h}_{\alpha-}\bar{h}_{\beta-} - \bar{h}_{\alpha-}\partial_r\bar{h}_{\beta-} + \bar{h}^{\gamma\delta}\partial_r\bar{h}_{\gamma\delta}\bar{h}_{\alpha-}\bar{h}_{\beta-}\right), \tag{4.22}$$

and we have

$$\frac{1}{2} n^\mu e^{-8\phi/3} F_{\mu\alpha} A_\beta = \frac{1}{2} r e^{5\phi/3}\left(2\partial_r\phi\bar{h}_{\alpha-}\bar{h}_{\beta-} + \partial_r\bar{h}_{\alpha-}\bar{h}_{\beta-}\right). \tag{4.23}$$

Thus,

$$\begin{aligned} \tau_{\alpha\beta} &= e^{-\phi/3}\bar{\pi}_{\alpha\beta} + \frac{1}{2} r e^{5\phi/3}(-\bar{h}_{\alpha-}\partial_r\bar{h}_{\beta-} + \bar{h}^{\gamma\delta}\partial_r\bar{h}_{\gamma\delta}\bar{h}_{\alpha-}\bar{h}_{\beta-}) \\ &\quad -\frac{1}{2}\left(-6 + (1+c_4\phi)A_\alpha A^\alpha + c_5(A_\alpha A^\alpha)^2 + (2c_4 - 4c_5 + 3)\phi^2\right) h_{\alpha\beta} \\ &= e^{-\phi/3}(\bar{\pi}_{\alpha\beta} + 3\bar{h}_{\alpha\beta}) - e^{5\phi/3}\bar{h}_{\alpha-}(\bar{\pi}_{\beta-} + 3\bar{h}_{\beta-}) \\ &\quad -\frac{1}{2}\left(6e^{\phi/3} - 6 + (1+c_4\phi)A_\alpha A^\alpha + c_5(A_\alpha A^\alpha)^2 + (2c_4 - 4c_5 + 3)\phi^2\right) h_{\alpha\beta}. \end{aligned} \tag{4.24}$$

The two expressions are thus clearly not manifestly the same. However, to compare them we should consider the behaviour at large r.

The asymptotically AdS solution has

$$\bar{h}_{\alpha\beta} = r^2\eta_{\alpha\beta} + \frac{1}{r^2}\bar{h}^{(1)}_{\alpha\beta}. \tag{4.25}$$

This implies that $h^{\alpha\beta} \sim r^{-2}$ except for $h^{--} \sim r^0$, and $\sqrt{-h} \sim r^4$, so for $\alpha \neq -$, finite contributions to T^α_β come from terms in $\tau_{\gamma\beta}$ which go like r^{-2}, and we can neglect any contribution which falls off more rapidly. We have $\bar{\pi}_{\alpha\beta} + 3\bar{h}_{\alpha\beta} \sim r^{-2}$, so the first term in (4.24) gives a finite contribution. For $\alpha \neq +$, $\bar{h}_{\alpha-} \sim r^{-2}$, so the second term can be neglected. For $\alpha = +$, however, the second term gives a potentially divergent contribution to the stress tensor. To calculate the last term in (4.24), it is useful to note that

$$A_\alpha A^\alpha = e^{8\phi/3}\bar{g}_{--}. \tag{4.26}$$

We then find

$$\begin{aligned} G &\equiv 6e^{\phi/3} - 6 + (1+c_4\phi)A_\alpha A^\alpha + c_5(A_\alpha A^\alpha)^2 + (2c_4 - 4c_5 + 3)\phi^2 \\ &= \left(\frac{83}{216} + \frac{5}{12}(c_4 - 4c_5)\right)\bar{g}^3_{--} + \ldots \sim \frac{1}{r^6}, \end{aligned} \tag{4.27}$$

where the dots denote terms of higher order in a Taylor expansion in $\bar{g}_{--}$. The last term hence can be neglected, except when $\alpha = \beta = +$ (as $h_{++} \sim r^4$ at large r).

Thus, for all the components where $\alpha \neq +$,

$$\tau_{\alpha\beta} = \bar{\pi}_{\alpha\beta} + 3\bar{h}_{\alpha\beta} + \mathcal{O}(r^{-4}). \tag{4.28}$$

For $\alpha = +$, $\beta \neq +$,

$$\tau_{+\beta} = -e^{5\phi/3}\bar{h}_{+-}(\bar{\pi}_{\beta-} + 3\bar{h}_{\beta-}) + e^{-\phi/3}(\bar{\pi}_{\beta+} + 3\bar{h}_{\beta+}) + \mathcal{O}(r^{-6}), \tag{4.29}$$

where the first term is order r^0, and the second term is order r^{-2}. For $\alpha = +$, $\beta = +$, there is an order r^{-2} term from the last term in (4.24), so

$$\tau_{++} = -e^{5\phi/3}\bar{h}_{+-}(\bar{\pi}_{+-} + 3\bar{h}_{+-}) + \mathcal{O}(r^{-2}). \tag{4.30}$$

Let us now consider the implications of this asymptotic behaviour for our non-relativistic stress tensor complex. The non-relativistic stress tensor complex defined in (4.9), (4.10), (4.11) is constructed from the components $T^\alpha_{\ \beta}$ with $\alpha \neq -$, so we are mainly interested in these. For $\alpha = i$,

$$T^i_{\ \beta} = \sqrt{-h}h^{i\gamma}\tau_{\gamma\beta} = e^{2\phi/3}r^2\tau_{i\beta} + \mathcal{O}(r^{-2}) = \bar{T}^i_{\ \beta} + \mathcal{O}(r^{-2}). \tag{4.31}$$

Similarly, for $\alpha = +$,

$$T^+_{\ \beta} = \sqrt{-h}h^{+\gamma}\tau_{\gamma\beta} = e^{2\phi/3}r^2\tau_{-\beta} + \mathcal{O}(r^{-2}) = \bar{T}^+_{\ \beta} + \mathcal{O}(r^{-2}). \tag{4.32}$$

Thus, for the components that contribute to our definition of the non-relativistic stress tensor complex, we find precise agreement with the definition of [23]. Note in particular that $\tau_{+\beta}$ will not affect these contributions, as $h^{+i}, h^{++} \sim r^{-6}$. Thus, our definition of the non-relativistic stress tensor complex and the definition proposed in [23] will agree on asymptotically Schrödinger solutions which are obtained by TsT transformation from a vacuum asymptotically AdS solution.

It is also interesting to consider what happens for the remaining components of the stress tensor, those with $\alpha = -$. We have

$$\begin{aligned} T^-_{\ \beta} = \sqrt{-h}h^{-\gamma}\tau_{\gamma\beta} &= \sqrt{-h}e^{2\phi/3}[\bar{h}^{-\gamma}\tau_{\gamma\beta} + \tau_{-\beta}] \\ &= \sqrt{-h}e^{2\phi/3}\bar{h}^{-\gamma}\left[e^{-\phi/3}(\bar{\pi}_{\gamma\beta} + 3\bar{h}_{\gamma\beta}) - \frac{1}{2}Gh_{\gamma\beta}\right] + \mathcal{O}(r^{-2}). \end{aligned} \tag{4.33}$$

There are two sources of potentially divergent contributions in this term, coming from the r^0 part in $\tau_{+\beta}$, and the r^{-2} part in $\tau_{-\beta}$. These both involve factors of $\bar{\pi}_{\beta-} + 3\bar{h}_{\beta-}$, and they cancel exactly to leave a finite result for this component of the stress tensor. The term involving G is negligible except for $\gamma = \beta = +$, so the components $T^-_{\ \beta}$ for $\beta \neq +$ will also agree with the definition of [23]. The component $T^-_{\ +}$, although finite, will not in general agree with the definition of [23].[5] However, this disagreement does not affect the physics. To make contact with a non-relativistic theory by light cone reduction, we are restricting to metrics which are independent of x^-. This implies that the $T^-_{\ \beta}$ drop out of the conservation equations; they are not part of the conserved currents associated with the restricted diffeomorphism freedom which preserves the manifest Killing symmetry along x^-. A disagreement in these components thus has no physical consequences for the non-relativistic dual.

In [25], it was shown that the action (4.2) satisfies $\delta S = 0$ for variations around the black hole solution (4.7) satisfying some rather restrictive boundary conditions. We

[5] We could choose the constants c_4, c_5 to make the $\bar{g}^3_{--}$ contribution to G vanish, and it would then agree. However, it is better to use this freedom instead to eliminate a divergence in s_ϕ, as we will shortly describe.

have shown that the stress tensor complex is finite for a large family of asymptotically Schrödinger solutions. Since our stress tensor is defined as the variation of the action with respect to a variation in the asymptotic boundary values of the frame fields, this implies that $\delta S = 0$ for any variation of the frame fields which does not change the asymptotic boundary values. However, the coefficients of matter field variations s_α, s_ϕ will still diverge for general asymptotically Schrödinger solutions (and in particular for the black hole solution (4.7)), so we still need to impose restrictive boundary conditions on the variations of the matter fields, as in [25]. We can make the divergent contribution to s_ϕ vanish by choosing the coefficients in the action so that $c_4 - 4c_5 + \frac{17}{3} = 0$, but we are still left with divergences in s_α. A more general understanding of these asymptotic boundary conditions is an interesting problem for the future.

5 Asymptotic perturbation analysis for Lifshitz

We want to show that the action (2.7) is finite on-shell and satisfies $\delta S = 0$ for a class of asymptotically Lifshitz spacetimes. Black hole solutions which asymptotically approach (2.4) were obtained in [27, 28], and we could consider the behaviour for these backgrounds. However, since the solutions are only known numerically, a direct analysis of these solutions is difficult and not very illuminating.[6] Instead, it is better to perform a general analysis of the equations of motion in the asymptotic region. Finding exact solutions of the equations of motion (2.2), (2.3) analytically is difficult. However, if the solution is asymptotically Lifshitz, it will be a small perturbation of (2.4) for sufficiently large r. Let us therefore study the solutions of the linearized equations of motion expanding around (2.4). This calculation will also be useful for obtaining two-point functions on the background (2.4), although we will not investigate this here. Note that the analysis of the constant scalar perturbations was also performed in [27–29]; perturbative analysis of related solutions was also performed in [37].

If we write the background as $g_{\mu\nu}$, A_μ and the perturbations as $h_{\mu\nu}$, a_μ, then the linearized equations are[7]

$$\nabla_\mu f^{\mu\nu} - \nabla_\mu (h^{\mu\lambda} F_\lambda{}^\nu) - \nabla_\mu h^{\beta\nu} F^\mu{}_\beta + \frac{1}{2} \nabla_\lambda h F^{\lambda\nu} = m^2 a^\nu \tag{5.1}$$

and

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \Lambda h_{\mu\nu} + \frac{1}{2} f_{\mu\lambda} F_\nu{}^\lambda + \frac{1}{2} f_{\nu\lambda} F_\mu{}^\lambda - \frac{1}{2} F_{\mu\lambda} F_{\nu\sigma} h^{\lambda\sigma} - \frac{1}{4} f_{\lambda\rho} F^{\lambda\rho} g_{\mu\nu} + \frac{1}{4} F_{\lambda\rho} F_\sigma{}^\rho h^{\lambda\sigma} g_{\mu\nu} \\ &\quad - \frac{1}{8} F_{\lambda\rho} F^{\lambda\rho} h_{\mu\nu} + \frac{1}{2} m^2 a_\mu A_\nu + \frac{1}{2} m^2 a_\nu A_\mu, \end{aligned} \tag{5.2}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and

$$R^{(1)}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} [\nabla_\lambda \nabla_\mu h_{\nu\sigma} + \nabla_\lambda \nabla_\nu h_{\mu\sigma} - \nabla_\mu \nabla_\nu h_{\lambda\sigma} - \nabla_\lambda \nabla_\sigma h_{\mu\nu}]. \tag{5.3}$$

[6]The only known analytic black hole solutions [28, 29] to (2.2), (2.3) have non-flat boundary.

[7]Note that $h_{\mu\nu}$ denotes the perturbation of the metric, and indices are raised and lowered with the background metric, so $h^{\mu\nu}$ is the perturbation of the metric with the indices raised, not the perturbation of the inverse metric. This differs from the convention in the discussion of the variation of the action, where $\delta h^{\mu\nu}$ is the variation of the inverse metric.

It is convenient to define

$$h_{tt} = -r^{2z}\hat{h}_{tt}, \qquad h_{ti} = -r^{2z}v_{1i} + r^2 v_{2i}, \qquad h_{ij} = r^2\hat{h}_{ij}, \tag{5.4}$$

$$a_t = \alpha r^z\left(\hat{a}_t + \frac{1}{2}\hat{h}_{tt}\right), \qquad a_i = \alpha r^z v_{1i}, \qquad a_r = \alpha\frac{\hat{a}_r}{r}. \tag{5.5}$$

We choose a Gaussian normal gauge, so $h_{r\mu} = 0$. In terms of a frame field, this definition corresponds to choosing the orthonormal frame

$$\begin{aligned} e^{(0)} &= r^z\hat{e}^{(0)} = r^z\left[\left(1 + \frac{1}{2}\hat{h}_{tt}\right)dt + v_{1i}dx^i\right], \\ e^{(i)} &= r\hat{e}^{(i)} = r\left[v_{2i}dt + \left(\delta^i{}_j + \frac{1}{2}\hat{h}^i{}_j\right)dx^j\right], \quad e^{(3)} = \frac{dr}{r}, \end{aligned} \tag{5.6}$$

and the vector field components in the orthonormal frame to be

$$A^M = \alpha(1 + \hat{a}_t)\delta^M_0 + \alpha\hat{a}_r\delta^M_3. \tag{5.7}$$

That is, we are partially fixing the freedom in the choice of frame (local Lorentz invariance) by choosing the frame vector $e^{(0)}$ to be parallel to the projection of the vector field A along the boundary at constant r.

For our spacetime to be asymptotically Lifshitz, we will at least require that the normalised perturbations $\hat{h}_{tt}$, v_{1i}, v_{2i}, $\hat{h}_{ij}$, $\hat{a}_t$ and $\hat{a}_r$ all vanish as $r \to \infty$. In terms of the frame fields, we are saying that a necessary condition for the spacetime to be asymptotically Lifshitz is that $\hat{e}^{(0)} \to dt$, $\hat{e}^{(i)} \to dx^i$, $A^M \to \alpha\delta^M_0$ as $r \to \infty$. We will be more precise about our boundary conditions once we have solved the linearised equations of motion.

One of our goals is to show that the action (2.7) is finite on-shell. In the linearised analysis, since the background solution has no vector-like parts in the spatial directions along the boundary and the action is a scalar, the action to linear order will only involve the scalar parts of the linearised perturbations. Furthermore, the integration over the boundary directions makes the value of the action depend only on the zero-momentum part of the perturbation. This also implies there is no contribution from $S_{\rm deriv}$ at linear order. There is a potential divergence in the action from the region at large r, where

$$A^2 = -\frac{2(z-1)}{z}(1 + 2\hat{a}_t), \quad F^2 = -4z(z-1)\left(1 + 2\hat{a}_t + \frac{2r}{z}\partial_r\hat{a}_t + \frac{r}{z}\partial_r\hat{h}_{tt}\right). \tag{5.8}$$

From the metric perturbation we have $\sqrt{-g} = r^{z+1}[1 + \frac{1}{2}(\hat{h}_{tt} + \hat{h}^i{}_i)]$, $\sqrt{-h} = r^{z+2}[1 + \frac{1}{2}(\hat{h}_{tt} + \hat{h}^i{}_i)]$, where $\hat{h}^i{}_i = \delta^{ij}\hat{h}_{ij}$,

$$R = -2z^2 - 4z - 6 - r^2\partial_r^2\hat{h}_{tt} - r^2\partial_r^2\hat{h}^i{}_i - (2z+3)r\partial_r\hat{h}_{tt} - (z+4)r\partial_r\hat{h}^i{}_i, \tag{5.9}$$

and

$$K = z + 2 + \frac{r}{2}\partial_r(\hat{h}_{tt} + \hat{h}^i{}_i). \tag{5.10}$$

Putting all of this into the action (2.7) for a region $r \leq r_0$ gives

$$\frac{S}{\mathrm{Vol}} = bulk + \int^{r_0} dr r^{z+1} \Big[-(z+2)(\hat{h}_{tt} + \hat{h}^i{}_i) + 2(z+2)(z-1)\hat{a}_t \\ -(z+4) r\partial_r(\hat{h}_{tt} + \hat{h}^i{}_i) + 2(z-1) r\partial_r \hat{a}_t - r^2 \partial_r^2 (\hat{h}_{tt} + \hat{h}^i{}_i) \Big] \\ + R^4 \Big[\hat{h}_{tt} + \hat{h}^i{}_i - 2(z-1)\hat{a}_t + r\partial_r(\hat{h}_{tt} + \hat{h}^i{}_i) \Big]_{r=r_0}, \tag{5.11}$$

where we have performed the integral over t, x, y and divided out the overall volume in these directions. We write *"bulk"* to indicate that we are only keeping track of the contribution to the action from the region at large r, where a linearised analysis is appropriate. In the next subsection, we will determine the asymptotic behaviour of these constant scalar perturbations, and show that the potential divergences in the contributions we have written explicitly in (5.11), coming from the region at large r, cancel to leave a finite result.

We also want to verify that the variation of the action vanishes on-shell for suitable boundary conditions on the variations. Our approach will be to verify this by showing that the stress tensor defined above is finite. The logic is that we can write the general on-shell variation of the action as in (3.6), with the variation δA^A restricted to a variation of δA^0 by our choice of frame. If the action has finite variations under variations which change the boundary data, the variation will then clearly vanish for any variations that do not change the boundary data (i.e., those which fall off fast enough at the boundary). We will show below as we analyse the perturbations that they give finite coefficients for variations of the boundary data, up to some subtleties in the variation of A^0. These subtleties are addressed in section 5.3, showing that the variation of the action vanishes for suitable asymptotically Lifshitz boundary conditions.

Consider therefore the calculation of the non-relativistic stress tensor complex defined in section 3 at linear order. Since the $s_{\alpha\beta}$ and s_β are already linear in terms of the perturbation, our prescription for the stress tensor complex reduces to

$$\mathcal{E} = -2r^{-2z} s_{tt} + \alpha r^{-z} s_t, \qquad \mathcal{E}_i = 2r^{-2} s_{ti} - \alpha r^{z-2} s_i, \tag{5.12}$$

$$\mathcal{P}_i = 2r^{-2z} s_{ti}, \qquad \Pi_{ij} = -2r^{-2} s_{ij}. \tag{5.13}$$

For the general perturbations, we should now include contributions from S_{deriv}. We discuss this part of the calculation in appendix A. The upshot of the analysis there is that the contributions from the derivative terms are suppressed relative to the contribution from the non-derivative part of the action, and as a result only make a finite contribution to the component $\mathcal{E}_y$ in the stress tensor complex, where they can be chosen to cancel divergences in the contributions from the non-derivative part of the action.

We can write the contribution from the remaining part of the action for our ansatz in a relatively simple form in terms of the asymptotic fields:

$$\begin{aligned}
\mathcal{E} &= -r^{z+2}\left[r\partial_r \hat{h}^i{}_i + \alpha^2(z\hat{a}_t + r\partial_r(\frac{1}{2}\hat{h}_{tt} + \hat{a}_t) - r^{-z}\partial_t\hat{a}_r)\right] + \mathcal{E}^{\text{deriv}}, \\
\mathcal{E}_i &= r^{z+2}\left[r\partial_r v_{2i} + \frac{(z-2)}{z}r^{2(z-1)}r\partial_r v_{1i} - \frac{2(z-1)}{z}r^{z-2}\partial_i\hat{a}_r\right] + \mathcal{E}_i^{\text{deriv}}, \\
\mathcal{P}_i &= r^{z+2}[-r\partial_r v_{1i} + r^{-2(z-1)}r\partial_r v_{2i}] + \mathcal{P}_i^{\text{deriv}}, \\
\Pi_{ij} &= -r^{z+2}[-r\partial_r\hat{h}_{tt}\delta_{ij} + r\partial_r(\hat{h}_{ij} - \delta_{ij}\hat{h}^k{}_k) + 2(z-1)\hat{a}_t\delta_{ij}] + \Pi_{ij}^{\text{deriv}}.
\end{aligned} \tag{5.14}$$

We will also want to evaluate

$$s_0 = -r^{-z}s_t = r^{z+2}\alpha\left[z\hat{a}_t + r\partial_r\left(\frac{1}{2}\hat{h}_{tt} + \hat{a}_t\right) - r^{-z}\partial_t\hat{a}_r\right] + s_0^{\text{deriv}}. \tag{5.15}$$

For completeness, we also note that

$$s_i = -r^{z+2}\alpha[r^z r\partial_r v_{1i} - \partial_i\hat{a}_r]. \tag{5.16}$$

In our linearised analysis, terms in the conservation equations involving the variation of the matter fields like the one appearing on the right-hand side of (3.4) will not appear, as both s_A and the derivative $\partial_\beta A^A$ are of linear order in the perturbation. We therefore expect our stress tensor complex to obey the conservation equations (3.1), and we will indeed find that the bulk equations of motion imply this conservation.

Finally, a note on the applicability of this linearised analysis. We can see from the form of the stress tensor that perturbations where the normalised fields fall off like $r^{-(z+2)}$ will be associated with finite contributions to some element of the stress tensor complex. Thus, if we have linear perturbations where the normalised fields fall off like $r^{-\frac{1}{2}(z+2)}$, then quadratic terms in these perturbations could make finite contributions to the stress tensor complex, and the linearised analysis we are performing would not be justified by the smallness of the fields in the asymptotic region; even to understand the asymptotic behaviour of a generic asymptotically Lifshitz solution with such falloffs could require a non-linear analysis.

5.1 Constant perturbations

Because the background is translation-invariant in t, x, y, we can decompose the perturbations into plane wave modes, and modes of different frequencies will not mix. We consider first the zero momentum part; perturbations which are constant in the boundary directions. These constant perturbations can be decomposed into scalar, vector and tensor parts:

$$h_{tt} = -r^{2z}f(r), \quad h_{ti} = -r^{2z}v_{1i}(r) + r^2 v_{2i}(r), \quad h_{ij} = r^2 k(r)\delta_{ij} + r^2 k_{ij}(r), \tag{5.17}$$

where

$$k_{ij}(r) = \begin{bmatrix} t_d(r) & t_o(r) \\ t_o(r) & -t_d(r) \end{bmatrix}, \tag{5.18}$$

and

$$a_t = \alpha r^z \left(j(r) + \frac{1}{2} f(r) \right), \quad a_i = \alpha r^z v_{1i}(r). \tag{5.19}$$

Note that a constant a_r component is forced to vanish by the equations of motion. As a consequence of the rotation invariance in the x, y plane in the background, the scalar, vector and tensor sectors do not mix.

We consider first the constant scalar perturbations. This will include as a special case the linearised version of the black hole solutions of [27, 28]. While this paper was in preparation, the perturbations in this scalar sector were analysed in [29], which also considers a background where the flat spatial slices are replaced by a sphere. Our results agree with this previous work, although direct comparison is not straightforward as we work in a different gauge.

The equations of motion for constant scalar modes reduce to

$$2r^2 j'' = (z+1)rf' - 4(z+1)rj' - (z+4)(2z-2)j, \tag{5.20}$$

$$\frac{1}{r^2}(z+1)(r^4 f')' = (z-1)(4z+2)rj' + (z-1)(4z^2+6z+8)j, \tag{5.21}$$

$$2(z+1)rk' = -(z+1)rf' - 2(z-1)rj' - (z-1)(2z-4)j. \tag{5.22}$$

The fact that these do not involve f, k undifferentiated reflects the freedom to shift coordinates by rescaling t, x, y.

For $z = 2$, the solution is

$$j(r) = -\frac{c_1 + c_2 \ln r}{r^4} + c_3, \tag{5.23}$$

$$f(r) = \frac{4c_1 - 5c_2 + 4c_2 \ln r}{12r^4} + 4c_3 \ln r + c_4, \tag{5.24}$$

$$k(r) = \frac{4c_1 + 5c_2 + 4c_2 \ln r}{24r^4} - 2c_3 \ln r + c_5. \tag{5.25}$$

We can set $c_4 = c_5 = 0$ by redefining the coordinates t, x, y. We should also set $c_3 = 0$ to satisfy the asymptotically Lifshitz boundary condition; that is, to ensure that the solution is small at large r, consistent with our assumption.

For $z \neq 2$, the solution is

$$j(r) = -\frac{(z+1)c_1}{(z-1)r^{z+2}} - \frac{(z+1)c_2}{(z-1)r^{\frac{1}{2}(z+2+\beta_z)}} + \frac{(z+1)c_3}{(z-1)r^{\frac{1}{2}(z+2-\beta_z)}}, \tag{5.26}$$

$$\begin{aligned} f(r) = {} & 4\frac{1}{(z+2)}\frac{c_1}{r^{z+2}} + 2\frac{(5z-2-\beta_z)}{(z+2+\beta_z)}\frac{c_2}{r^{\frac{1}{2}(z+2+\beta_z)}} \\ & -2\frac{(5z-2+\beta_z)}{(z+2-\beta_z)}\frac{c_3}{r^{\frac{1}{2}(z+2-\beta_z)}} + c_4, \end{aligned} \tag{5.27}$$

$$\begin{aligned} k(r) = {} & 2\frac{1}{(z+2)}\frac{c_1}{r^{z+2}} - 2\frac{(3z-4-\beta_z)}{(z+2+\beta_z)}\frac{c_2}{r^{\frac{1}{2}(z+2+\beta_z)}} \\ & +2\frac{(3z-4+\beta_z)}{(z+2-\beta_z)}\frac{c_3}{r^{\frac{1}{2}(z+2-\beta_z)}} + c_5, \end{aligned} \tag{5.28}$$

where $\beta_z^2 = 9z^2 - 20z + 20 = (z+2)^2 + 8(z-1)(z-2)$.

Let us use these solutions to be more precise about the asymptotically Lifshitz boundary conditions. We see that there are constant modes in f and k, which can be interpreted as changes in the boundary data for the metric. For j, by contrast, there is no constant mode for general z. The slowest falloff in j is given by the mode parametrized by c_3, which falls off as $r^{-\frac{1}{2}(z+2-\beta_z)}$ (it is constant in the special case $z = 2$). Thus, for $1 \leq z < 2$, we want to interpret the mode parametrised by c_3 as the boundary data for the vector field. To fix this boundary data, we need to require that $r^{\frac{1}{2}(z+2-\beta_z)}(A^M - \alpha\delta_0^M)$ vanishes as $r \to \infty$.[8] For $z \geq 2$, this mode produces terms in f and k which grow at large r, and hence violate the boundary conditions for those fields. It is therefore not clear whether we can think of this as boundary data for the vector field in this case. For $z \geq 2$, we will simply impose the boundary condition that $A^M - \alpha\delta_0^M$ vanishes as $r \to \infty$. We therefore adopt as our definition of asymptotically Lifshitz boundary conditions that $\hat{h}_{tt}$, v_{1i}, v_{2i}, $\hat{h}_{ij}$ and $\hat{a}_t$ vanish as $r \to \infty$, and that for $1 \leq z < 2$, $r^{\frac{1}{2}(z+2-\beta_z)}\hat{a}_t \to 0$ as $r \to \infty$.

We thus have a two-parameter family of solutions in this constant scalar sector, parametrized by c_1, c_2. In [29], the energy for these solutions was evaluated by background subtraction, and they found that for $z \leq 2$, they needed to set $c_2 = 0$ as well to have a finite energy. We will see below that with our definition of the boundary energy density, we get finite results for any z without further restricting the solutions.[9] The divergences found in [29] are due to using an action which does not include the surface terms necessary to ensure the action is finite on-shell. In the cases $z \leq 2$, the asymptotically Lifshitz solution approaches the background too slowly at large r for these surface terms to cancel out in the background subtraction calculation. A similar failure of background subtraction occurs for the Schrödinger case [25, 37].

We first want to use these scalar modes to evaluate the on-shell value of the action (5.11). For $z = 2$, we find

$$\frac{S}{\mathrm{Vol}} = bulk + \frac{2}{3}c_2, \tag{5.29}$$

and for $z \neq 2$, we find

$$\frac{S}{\mathrm{Vol}} = bulk + \frac{2(z+1)(z-2)}{(z+2)}c_1. \tag{5.30}$$

Thus, we see that the potential divergences from the region at large r cancel, to leave a finite answer for this part of the action. We have not explicitly considered the contribution to the action from the interior of the spacetime, so there could still be a divergence there, but this is unlikely. In appendix B, we consider the action for a set of static Euclidean black hole solutions, and see explicitly that it is finite.

We next consider the contribution to the stress tensor from these modes, which gives

$$\mathcal{E} = -r^{z+2}\left[2r\partial_r k + \alpha^2\left(zj + r\partial_r j + \frac{1}{2}r\partial_r f\right)\right] = \begin{cases} \frac{4(z-2)}{z}c_1 & \text{for } z \neq 2 \\ \frac{4c_2}{3} & \text{for } z = 2. \end{cases} \tag{5.31}$$

[8] Note that this implies, surprisingly, that the boundary data are subleading compared to the background value for A^M. For general z, the allowed changes in the boundary data for the massive vector do not change the $\alpha\delta_0^M$ term, but add a term falling off like $r^{-\frac{1}{2}(z+2-\beta_z)}$ to it. Apart from the subtraction of the $\alpha\delta_0^M$ term, this is like the boundary condition for a massive vector in the relativistic case.

[9] In fact, for constant modes, we have a finite energy even if we allow $c_3 \neq 0$.

Note that the separate contributions are divergent (logarithmically for $z = 2$), but the combination is finite. We have

$$\Pi_{ij} = -2r^{-2}s_{ij} = -2r^{z+2}\left[(z-1)j - \frac{r}{2}\partial_r f - \frac{r}{2}\partial_r k\right]\delta_{ij} = \begin{cases} 2(z-2)c_1\delta_{ij} & \text{for } z \neq 2 \\ \frac{4c_2}{3}\delta_{ij} & \text{for } z = 2. \end{cases} \quad (5.32)$$

For the black hole solutions [27, 28], only these scalar modes are turned on, so this gives the thermal stress tensor dual to the black hole solution. The bulk black hole can be used to relate the energy density, which is an arbitrary constant of integration in our asymptotic analysis, to the temperature. Note that this thermal stress tensor satisfies the equation of state $z\mathcal{E} = \delta^{ij}\Pi_{ij}$ required by the anisotropic scaling symmetry.

For the vector modes, the equations are

$$r^2 v_{1i}'' + (2z+1) r v_{1i}' + z r^{-2(z-1)} r v_{2i}' = 0, \quad (5.33)$$

$$r^2 v_{2i}'' + 5 r v_{2i}' + (z-2) r^{2(z-1)} r v_{1i}' = 0. \quad (5.34)$$

For $z \neq 4$ the solutions are

$$\begin{aligned} v_{1i}(r) &= c_{1i} + \frac{c_{2i}}{r^{z+2}} + \frac{c_{3i}}{r^{3z}}, \\ v_{2i}(r) &= \frac{(z^2-4)}{z(z-4)} c_{2i} r^{z-4} + \frac{3z}{(z+2)}\frac{c_{3i}}{r^{z+2}} + c_{4i}, \end{aligned} \quad (5.35)$$

and for $z = 4$ we have

$$\begin{aligned} v_{1i}(r) &= c_{1i} + \frac{c_{2i}}{r^6} + \frac{c_{3i}}{r^{12}}, \\ v_{2i}(r) &= 3\ln(r) c_{2i} + 2\frac{c_{3i}}{r^6} + c_{4i}. \end{aligned} \quad (5.36)$$

These give contributions to the stress tensor complex which are

$$\mathcal{E}_i = r^{z+2}\left[r\partial_r v_{2i} + \frac{(z-2)}{z} r^{2(z-1)} r\partial_r v_{1i}\right] = -6(z-1)c_{3i}, \quad (5.37)$$

and

$$\mathcal{P}_i = r^{z+2}[-r\partial_r v_{1i} + r^{-2(z-1)} r\partial_r v_{2i}] = \frac{2(z-1)(z+2)}{z} c_{2i}. \quad (5.38)$$

In the vector solutions, c_{1i} is a pure gauge mode corresponding to shifting $t \to t + c_{1i}x^i$, and c_{4i} is a pure gauge mode corresponding to shifting $x^i \to x^i + c_{4i}t$. The contribution of c_{2i} to v_{2i} falls off more slowly than $r^{-(z+2)/2}$ for $z > 2$, so we would expect a linearised analysis to be insufficient to correctly extract the boundary stress tensor for a generic asymptotically Lifshitz geometry for $z \geq 2$. For $z \geq 4$, we need to set the coefficient c_{2i} to zero to satisfy the boundary conditions on v_{2i}. This is a further restriction on the space of allowed solutions, which makes the space of allowed solutions (at least in the linearised approximation) two dimensions smaller. This restriction sets $\mathcal{P}_i = 0$ for all asymptotically Schrödinger solutions with $z \geq 4$. It would be very interesting to understand this restriction from the dual field theory point of view. It would also be interesting to see if one can construct solutions with a non-zero boost that are physically acceptable and at the same time have $c_{2i} = 0$.

For the tensor modes, the equations are

$$r^2 t_d'' + (z+3) r t_d' = 0, \tag{5.39}$$

$$r^2 t_o'' + (z+3) r t_o' = 0, \tag{5.40}$$

and the solutions are

$$t_d(r) = t_{d1} + \frac{t_{d2}}{r^{z+2}}, \quad t_o(r) = t_{o1} + \frac{t_{o2}}{r^{z+2}}. \tag{5.41}$$

The constant terms are pure gauge, corresponding to relative scaling and rotation of the x, y coordinates respectively. The tensor modes source

$$\Pi_{ij} = -r^{z+2} r \partial_r k_{ij}, \tag{5.42}$$

which gives

$$\Pi_{xy} = (z+2) t_{o2}, \quad \Pi_{xx} = -\Pi_{yy} = (z+2) t_{d2}. \tag{5.43}$$

Since all of the constant perturbation modes give constant components for the stress tensor, the conservation equations are trivially satisfied.

In summary, for the constant perturbations, we have an eight-parameter family of solutions of the linearised equations of motion satisfying our asymptotic boundary conditions. Seven of these parameters correspond to the independent components of the stress tensor complex; there is an additional linearised solution in the scalar sector which does not contribute to the stress tensor at this order. For $1 \leq z < 2$, we have tightened our boundary conditions by setting $c_3 = 0$ even though this mode does not grow asymptotically. For $z \geq 4$, we must set $\mathcal{P}_i = 0$ to satisfy our boundary conditions, and we have a six-parameter family of solutions in the bulk.

5.2 General perturbations

Taking a particular plane wave mode, we can again decompose the perturbation into scalar and vector parts (for non-zero momentum, there is no transverse tracefree tensor in two dimensions). We simplify the analysis by using the rotation invariance to take the momentum to lie only along the x direction. Then we can write[10]

$$h_{tt} = -r^{2z} f(r) e^{i\omega t + ikx}, \qquad h_{tx} = k[-r^{2z} s_1(r) + r^2 s_2(r)] e^{i\omega t + ikx}, \tag{5.44}$$

$$h_{ty} = [-r^{2z} v_1(r) + r^2 v_2(r)] e^{i\omega t + ikx}, \tag{5.45}$$

$$h_{xx} = r^2 \left(k_L(r) + k^2 k_T(r)\right) e^{i\omega t + ikx}, \qquad h_{yy} = r^2 \left(k_L(r) - k^2 k_T(r)\right) e^{i\omega t + ikx}, \tag{5.46}$$

$$h_{xy} = r^2 k v_3(r) e^{i\omega t + ikx}, \tag{5.47}$$

and

$$a = \alpha r^z e^{i\omega t + ikx} \left[\left(j(r) + \frac{1}{2} f(r) \right) dt + k s_1(r) dx + v_1(r) dy + i \frac{p(r)}{r^{z+1}} dr \right]. \tag{5.48}$$

[10] To avoid cluttering the notation, we will not introduce subscripts ω, k on the functions in this ansatz to denote the mode we are considering. We hope this will not lead to confusion.

The functions v_1, v_2, v_3 represent divergence-free vector excitations, while the other functions are scalars or scalar-derived vectors with respect to the rotational symmetry in the x, y plane. The scalar modes and vector modes decouple, so we can analyse them separately.

5.2.1 Scalar modes

For the scalar part, one can bring the equations of motion to a nicer form by rescaling $s_1 \to \omega s_1$, $s_2 \to \omega s_2$. The function $p(r)$ appearing in a_r is determined algebraically,

$$p(r) = \frac{\omega}{4(z-1)r^z}\left[-2rk_L' + 2(z-1)k_L + k^2(rs_2' - r^{2z-1}s_1' - 2(z-1)s_2)\right] \tag{5.49}$$

and using this to eliminate $p(r)$, the remaining equations of motion for the modes in the scalar sector are

$$\begin{aligned} -2(z-1)rj' + (z+1)rf' - 6z(z-1)j &= -\frac{k^2}{r^2}(-2(z+1)r^3k_T' + k_L + f - k^2k_T) \\ &\quad -\frac{\omega^2}{r^{2z}}(rk_L' + 2(z+1)r^3(s_2' - r^{2(z-1)}s_1') - (z+1)k_L) \\ &\quad -\frac{\omega^2k^2}{2r^{2z}}(-rs_2' + r^{2z-1}s_1' + 2(z+1)s_2 - 4r^{2(z-1)}s_1), \end{aligned} \tag{5.50}$$

$$\begin{aligned} r^2f'' - (2z-3)rf' + 8(z-1)^2j &= \frac{k^2}{r^2}(-2(2z+1)r^3k_T' + k_L + 2f - k^2k_T) \\ &\quad +\frac{\omega^2}{r^{2z}}(2(2z+1)r^3(s_2' - r^{2(z-1)}s_1') - 4k_L) + \frac{4k^2\omega^2}{r^{2z}}(s_2 - r^{2(z-1)}s_1), \end{aligned} \tag{5.51}$$

$$rk_L' + rf' - 2(z-1)j = k^2rk_T' + \omega^2(rs_1' - r^{-2(z-1)}rs_2'), \tag{5.52}$$

$$r^4k_T'' + (z+3)r^3k_T' - \frac{1}{2}f = -\omega^2\left[s_1 + r^{-2(z-1)}(k_T - s_2)\right], \tag{5.53}$$

$$\begin{aligned} &-2r^4s_2'' + 2r^{2z+2}s_1'' + 2(z-5)r^3s_2' \\ &\quad +2(z+3)r^{2z+1}s_1' - 2rf' + 4(z-1)j - 2(z-2)k_L \\ &\qquad = k^2(rs_2' - r^{2z-1}s_1' - 2rk_T' - 2(z-1)s_2 + 2k_T) + 2\omega^2(r^{-2z+3}s_2' - rs_1'), \end{aligned}$$

$$\begin{aligned} &k^2(-2rf' - 4r^3k_T' - 2(z-2)r^{2z+1}s_1' - 2zr^3s_2' + 2f + 4(z-1)j - 2(z-2)k_L) \\ &\quad +\frac{\omega^2}{r^{2(z-1)}}(4r^3s_2' - 4r^{2z+1}s_1' - 4k_L) \\ &\quad +\frac{k^2\omega^2}{r^{2z}}(2r^{2z+1}s_1' - 2r^3s_2' + 4r^2s_2 - 4r^{2z}s_1) \\ &\quad +k^4(r^{2z-1}s_1' - rs_2' + 2rk_T' + 2(z-1)s_2 - 2k_T) = 0. \end{aligned} \tag{5.54}$$

For the scalar modes, the non-zero contributions to the boundary stress tensor complex are

$$\begin{aligned} \mathcal{E} &= -r^{z+2}\left[2r\partial_rk_L + \alpha^2(zj + r\partial_r(\frac{1}{2}f + j) + \omega r^{-z}p)\right]e^{i\omega t + ikx} + \mathcal{E}^{\text{deriv}}, \\ \mathcal{E}_x &= r^{z+2}\left[k\omega r\partial_rs_2 + k\omega\frac{(z-2)}{z}r^{2(z-1)}r\partial_rs_1 + \frac{2(z-1)}{z}kr^{z-2}p\right]e^{i\omega t + ikx} + \mathcal{E}_x^{\text{deriv}}, \\ \mathcal{P}_x &= r^{z+2}[-k\omega r\partial_rs_1 + k\omega r^{-2(z-1)}r\partial_rs_2]e^{i\omega t + ikx} + \mathcal{P}_x^{\text{deriv}}, \\ \Pi_{xx} &= -r^{z+2}[-r\partial_rf + r\partial_r(-k_L + k^2k_T) + 2(z-1)j]e^{i\omega t + ikx} + \Pi_{xx}^{\text{deriv}}, \\ \Pi_{yy} &= -r^{z+2}[-r\partial_rf + r\partial_r(-k_L - k^2k_T) + 2(z-1)j]e^{i\omega t + ikx} + \Pi_{yy}^{\text{deriv}}. \end{aligned} \tag{5.55}$$

We see in appendix A that the derivative terms make a vanishing contribution for the scalar modes.

The full stress tensor is conserved by virtue of the bulk equations of motion: for the terms in (5.55), the xr component of Einstein's equation gives the conservation equation $\omega\mathcal{P}_x + k\Pi_{xx} = 0$, and a combination of the tr component of Einstein's equation and the r component of the massive vector equation gives the conservation equation $\omega\mathcal{E} + k\mathcal{E}_x = 0$.

We solve (5.50)–(5.54) by writing each of the functions in a power series in ω, k. If we denote the functions collectively by F, we have $F = \sum_{l,m} k^{2l}\omega^{2m}F^{(l,m)}$. The equations for the $(0,0)$ part of the functions are obtained by taking the $k^0\omega^0$ part of (5.50)–(5.53) and the k^2 and ω^2 parts of (5.54) (which imply the $k^0\omega^0$ part of (5.54)). The equations are then

$$
\begin{aligned}
-2(z-1)rj^{(0,0)\prime} + (z+1)rf^{(0,0)\prime} - 6z(z-1)j^{(0,0)} &= 0, && (5.56)\\
r^2 f^{(0,0)\prime\prime} - (2z-3)rf^{(0,0)\prime} + 8(z-1)^2 j^{(0,0)} &= 0, && (5.57)\\
rk_L^{(0,0)\prime} + rf^{(0,0)\prime} - 2(z-1)j^{(0,0)} &= 0, && (5.58)\\
r^4 k_T^{(0,0)\prime\prime} + (z+3)r^3 k_T^{(0,0)\prime} - \frac{1}{2}f^{(0,0)} &= 0, && (5.59)\\
2(z-1)r^3 s_2^{(0,0)\prime} + 2r^3 k_T^{(0,0)\prime} + rf^{(0,0)\prime} - f^{(0,0)} - 2(z-1)j^{(0,0)} &= 0, && (5.60)\\
2(z-1)r^{2(z-1)}r^3 s_1^{(0,0)\prime} + 2r^3 k_T^{(0,0)\prime} + rf^{(0,0)\prime} - f^{(0,0)} + 2(z-1)k_L^{(0,0)} - 2(z-1)j^{(0,0)} &= 0. && (5.61)
\end{aligned}
$$

The solution of the first two equations for f, j is

$$
\begin{aligned}
j^{(0,0)} &= \frac{(z+1)d_2}{(z-1)r^{\frac{1}{2}(z+2+\beta_z)}} + \frac{(z+1)d_3}{(z-1)r^{\frac{1}{2}(z+2-\beta_z)}}, && (5.62)\\
f^{(0,0)} &= -\frac{2(5z-2-\beta_z)}{(z+2+\beta_z)}\frac{d_2}{r^{\frac{1}{2}(z+2+\beta_z)}} - \frac{2(5z-2+\beta_z)}{(z+2-\beta_z)}\frac{d_3}{r^{\frac{1}{2}(z+2-\beta_z)}} + d_4 && (5.63)
\end{aligned}
$$

for $z \neq 2$, and

$$
j^{(0,0)} = \frac{3d_2}{r^4} + d_3, \quad f^{(0,0)} = -\frac{d_2}{r^4} + 4d_3 \ln r + d_4 \tag{5.64}
$$

for $z = 2$. In the other functions, in addition to the terms sourced by these modes, there is an arbitrary constant term in k_L, and a solution for $k_T^{(0,0)} = d_5 + d_1 r^{-z-2}$. These source terms in s_1 and s_2, which also have arbitrary constant terms. Set all the constant terms to zero to satisfy the asymptotic boundary conditions, and also set $d_3 = 0$ as in the discussion of the constant modes.

We are then left with two solutions of the coupled system: the first is[11]

$$j^{(0,0)} = \frac{(z+1)c_1}{(z-1)r^{\frac{1}{2}(z+2+\beta_z)}}, \qquad f^{(0,0)} = -\frac{2(5z-2-\beta_z)}{(z+2+\beta_z)}\frac{c_1}{r^{\frac{1}{2}(z+2+\beta_z)}}, \tag{5.65}$$

$$k_L^{(0,0)} = \frac{2(3z-4-\beta_z)}{(z+2+\beta_z)}\frac{c_1}{r^{\frac{1}{2}(z+2+\beta_z)}}, \tag{5.66}$$

$$k_T^{(0,0)} = -\frac{(z+1)(5z-2-\beta_z)}{2(z+2+\beta_z)(z^2-3z+4+\beta_z)}\frac{c_1}{r^{\frac{1}{2}(z+6+\beta_z)}}, \tag{5.67}$$

$$s_1^{(0,0)} = s_1^{\text{coeff}}\frac{c_1}{r^{2z}r^{\frac{1}{2}(z+2+\beta_z)}}, \qquad s_2^{(0,0)} = s_2^{\text{coeff}}\frac{c_1}{r^2 r^{\frac{1}{2}(z+2+\beta_z)}} \tag{5.68}$$

for $z \neq 2$, and

$$j^{(0,0)} = \frac{3c_1}{r^4}, \qquad f^{(0,0)} = -\frac{c_1}{r^4}, \tag{5.69}$$

$$k_L^{(0,0)} = -\frac{c_1}{2r^4}, \qquad k_T^{(0,0)} = -\frac{c_1}{24r^6}, \tag{5.70}$$

$$s_1^{(0,0)} = -\frac{3c_1}{32r^8}, \qquad s_2^{(0,0)} = -\frac{c_1}{24r^6} \tag{5.71}$$

for $z = 2$.

This first solution will not give a contribution to the stress tensor. For $z > 2$, its contribution is a negative power of r, so it vanishes in any case. For $z \leq 2$, the contribution of this leading-order part is a non-negative power of r, but an explicit calculation shows that the coefficient vanishes, as for the constant perturbations. As $\beta_z - (z+2) > -1/2$, the first subleading piece, which is suppressed by k^2/r^2 relative to the leading pieces, will always give a negative power of r, so we do not need to compute it. Thus, the mode parametrized by c_1 makes zero contribution to the stress tensor complex.

The other solution of the leading-order equations satisfying our boundary conditions is

$$k_T^{(0,0)} = -\frac{c_2}{r^{z+2}}, \quad s_1^{(0,0)} = \frac{(z+2)c_2}{3z(z-1)r^{2(z-1)}r^{z+2}}, \quad s_2^{(0,0)} = \frac{c_2}{(z-1)r^{z+2}}. \tag{5.72}$$

This will make a finite contribution to the stress tensor complex. To calculate it fully, we need to first calculate some of the higher-order terms in our expansion.

Next we consider the solution for the functions $F^{(1,0)}$. The equations determining these functions will be the k^2 components of (5.50)–(5.54) and the k^4 component of (5.54). These equations are

$$\begin{aligned}
-2(z-1)rj^{(1,0)\prime} + (z+1)rf^{(1,0)\prime} - 6z(z-1)j^{(1,0)} &= -\frac{1}{r^2}(-2(z+1)r^3 k_T^{(0,0)\prime} + k_L^{(0,0)} + f^{(0,0)}),\\
r^2 f^{(1,0)\prime\prime} - (2z-3)rf^{(1,0)\prime} + 8(z-1)^2 j^{(1,0)} &= \frac{1}{r^2}(-2(2z+1)r^3 k_T^{(0,0)\prime} + k_L^{(0,0)} + 2f^{(0,0)}),\\
rk_L^{(1,0)\prime} + rf^{(1,0)\prime} - 2(z-1)j^{(1,0)} &= rk_T^{(0,0)\prime},\\
r^4 k_T^{(1,0)\prime\prime} + (z+3)r^3 k_T^{(1,0)\prime} - \frac{1}{2}f^{(1,0)} &= 0,
\end{aligned} \tag{5.73}$$

[11] We are introducing new constants c_i here to parametrize the independent solutions which satisfy the asymptotic boundary conditions. The s_1^{coeff}, s_2^{coeff} are unimportant but complicated numerical factors, so we do not write them explicitly.

JHEP09(2009)009

$$
\begin{aligned}
&-2r^4 s_2^{(1,0)''} + 2r^{2z+2} s_1^{(1,0)''} + 2(z-5)r^3 s_2^{(1,0)'} + 2(z+3)r^{2z+1} s_1^{(1,0)'} \\
&\quad -2rf^{(1,0)'} + 4(z-1)j^{(1,0)} - 2(z-2)k_L^{(1,0)} \\
&\qquad\quad = (rs_2^{(0,0)'} - r^{2z-1}s_1^{(0,0)'} - 2rk_T^{(0,0)'} - 2(z-1)s_2^{(0,0)} + 2k_T^{(0,0)}) \\
&-2rf^{(1,0)'} - 4r^3 k_T^{(1,0)'} - 2(z-2)r^{2z+1}s_1^{(1,0)'} - 2zr^3 s_2^{(1,0)'} \\
&\quad +2f^{(1,0)} + 4(z-1)j^{(1,0)} - 2(z-2)k_L^{(1,0)} \\
&\qquad\quad = -(r^{2z-1}s_1^{(0,0)'} - rs_2^{(0,0)'} + 2rk_T^{(0,0)'} + 2(z-1)s_2^{(0,0)} - 2k_T^{(0,0)}).
\end{aligned}
$$

This system will have a homogeneous solution of the same form as the solution of the $F^{(0,0)}$ equations; we can absorb that into the $F^{(0,0)}$ solution by a redefinition of c_1, c_2. We will absorb all homogeneous solutions of the same form at higher orders in the same way, promoting these constants to arbitrary functions of k, ω. Because the equations for s_1 and s_2 are different, there is an additional homogeneous solution which did not appear in the $F^{(0,0)}$ solutions. This is

$$
s_1^{(1,0)} = \frac{c_3}{r^{z+2}}, \quad s_2^{(1,0)} = \frac{(z^2-4)}{z(z-4)} c_3 r^{z-4}. \tag{5.74}
$$

As in the constant perturbations, we must set $c_3 = 0$ for $z \geq 4$ to satisfy the asymptotic boundary condition that $s_2 \to 0$ as $r \to \infty$.

In addition to the homogeneous solutions, we will have particular integrals for the sources from the $F^{(0,0)}$ solutions. As we have said above, there will be non-trivial particular integrals for the solution parametrized by c_1, but they do not contribute to the stress tensor, so we will not calculate them explicitly. For the solution parametrized by c_2, a particular integral for $z \neq 2$ is

$$
f^{(1,0)} = \frac{2c_2}{(z-2)r^{z+2}}, \qquad j^{(1,0)} = -\frac{(z+2)(z+1)c_2}{2(z-2)(z-1)r^{z+2}}, \tag{5.75}
$$

$$
k_L^{(1,0)} = \frac{c_2}{(z-2)r^{z+2}}, \qquad k_T^{(1,0)} = \frac{c_2}{2(z+4)(z-2)r^{z+4}}, \tag{5.76}
$$

$$
s_1^{(1,0)} = \frac{3c_2}{2(z-1)(z-2)(3z+2)r^{2(z-1)}r^{z+4}}, \qquad s_2^{(1,0)} = -\frac{(z-4)c_2}{2z(z-1)(z-2)(z+4)r^{z+4}}. \tag{5.77}
$$

For $z = 2$, a particular integral is

$$
j^{(1,0)} = -\frac{9c_2 \ln r}{r^4}, \qquad f^{(1,0)} = \frac{3c_2 \ln r}{r^4} + \frac{c_2}{4r^4}, \qquad k_L^{(1,0)} = \frac{3c_2 \ln r}{2r^4} - \frac{c_2}{8r^4}, \tag{5.78}
$$

$$
k_T^{(1,0)} = \frac{c_2 \ln r}{8r^6} + \frac{3c_2}{32r^6}, \qquad s_1^{(1,0)} = \frac{9c_2 \ln r}{32r^8} - \frac{93c_2}{256r^8}, \qquad s_2^{(1,0)} = \frac{c_2 \ln r}{8r^6} - \frac{13c_2}{32r^6}. \tag{5.79}
$$

Only the terms in (5.75) or (5.78) contribute to the stress tensor. This solution will lead to further contributions in the higher $F^{(l,m)}$, but they are suppressed by further powers of r, so they do not contribute to the stress tensor complex. We can therefore evaluate the contribution for this mode,

$$\mathcal{E} = \frac{2(z+2)}{z} c_2 k^2 e^{i\omega t + ikx}, \qquad \mathcal{E}_x = -\frac{2(z+2)}{z} c_2 k\omega e^{i\omega t+ikx}, \tag{5.80}$$

$$\mathcal{P}_x = 0, \qquad \Pi_{xx} = 0, \qquad \Pi_{yy} = 2(z+2) c_2 k^2 e^{i\omega t+ikx}. \tag{5.81}$$

The conservation equation $\omega\mathcal{E} + k\mathcal{E}_x = 0$ and the trace condition $z\mathcal{E} = \delta^{ij}\Pi_{ij}$ are satisfied as required.

We can carry on and calculate the equations of motion for the $F^{(0,1)}$ functions. The relevant equations are the ω^2 parts of (5.50)–(5.53) and the $k^2\omega^2$ and ω^4 parts of (5.54). As a result, the homogeneous solutions will be exactly the same as for the $F^{(0,0)}$, and we are only interested in the particular integrals which can contribute to the stress tensor. The only relevant terms are the ones proportional to c_3. The only source term from the $F^{(1,0)}$ functions in the equations for the $F^{(0,1)}$ functions is in the equation obtained from the $k^2\omega^2$ part of (5.54),

$$\begin{aligned} -2rf^{(0,1)\prime} - 4r^3 k_T^{(0,1)\prime} - 2(z-2)r^{2z+1}s_1^{(0,1)\prime} - 2zr^3 s_2^{(0,1)\prime} \\ +2f^{(0,1)} + 4(z-1)j^{(0,1)} - 2(z-2)k_L^{(0,1)} \\ = -\frac{1}{r^{2(z-1)}}(4r^3 s_2^{(1,0)\prime} - 4r^{2z+1}s_1^{(1,0)\prime} - 4k_L^{(1,0)}). \end{aligned} \tag{5.82}$$

A particular integral which satisfies the full set of equations for the $F^{(0,1)}$ functions is

$$k_T^{(0,1)} = -\frac{2(z-1)c_3}{zr^{z+2}}. \tag{5.83}$$

At higher orders, there will be no new homogeneous solutions. The homogeneous part of the equations for $F^{(l,m)}$ is the same as $F^{(1,0)}$ for $l \neq 0$, and is the same as $F^{(0,0)}$ for $l = 0$.[12] Thus, we can absorb the homogeneous solution into a redefinition of c_1, c_2, c_3. As for the particular integrals, we have obtained all the terms involving c_1, c_2 which can affect the stress tensor; higher terms are suppressed. For the solutions involving c_3, there is no source term in (5.50)–(5.53) for the functions $f^{(2,0)}, j^{(2,0)}, k_L^{(2,0)}, k_T^{(2,0)}$. The solution is therefore simply

$$s_1^{(2,0)} = -\frac{(z+2)^2}{2z(z^2-16)}\frac{c_3}{r^{z+4}}, \qquad s_2^{(2,0)} = -\frac{(z+2)}{2z(z-6)}c_3 r^{z-6}. \tag{5.84}$$

For the functions $F^{(1,1)}$, there is in principle a source term in (5.50)–(5.52), but it involves the combination

$$rk_T^{(0,1)\prime} + rs_1^{(1,0)\prime} - r^{-2(z-1)}rs_2^{(1,0)\prime}, \tag{5.85}$$

which vanishes by virtue of the equation of motion for $k_T^{(0,1)}$, (5.82). Thus, the particular integral will only involve $k_T^{(1,1)}$, $s_1^{(1,1)}$ and $s_2^{(1,1)}$, with powers of r such that the resulting particular integral makes no contribution to the stress tensor.

[12] The equations of motion for $F^{(0,m)}$ are in general the ω^{2m} part of (5.50)–(5.53) and the $k^2\omega^{2m}$ and the ω^{2m+2} parts of (5.54). One can check that in general the ω^{2m+2} part of (5.54) together with the ω^{2m} part of (5.52) imply the ω^{2m} part of (5.54).

Considering the stress tensor for the solutions proportional to c_3, we see that there are potentially divergent contributions to $\mathcal{E}_x$ coming from $s_1^{(l,0)}$ and $s_2^{(l,0)}$ for $l < z$. However, for this mode (recall again that there is no source for k_L at this order and so no k_L in the formula below)

$$\mathcal{E}_x = r^{z+2}\omega k \left[r s_2' + \frac{(z-2)}{z} r^{2z-1} s_1' + \frac{k^2}{2zr^2}(r s_2' - r^{2z-1} s_1' - 2(z-1)s_2) \right] e^{i\omega t + ikx}, \quad (5.86)$$

and this will vanish by virtue of the ω^0 part of (5.54). This is not surprising; having learnt that there are no divergent contributions to $\mathcal{E}$, a divergent contribution to $\mathcal{E}_x$ would be incompatible with the energy conservation equation. We can see this explicitly at the first two orders in k^2 using the $s_i^{(1,0)}$ and $s_i^{(2,0)}$ calculated above.

The contribution to the stress tensor from the solution parametrized by c_3 is then

$$\mathcal{P}_x = 2\frac{(z-1)(z+2)}{z}\omega k^3 c_3 e^{i\omega t + ikx}, \quad \Pi_{xx} = -\Pi_{yy} = -2\omega^2 k^2 \frac{(z-1)(z+2)}{z} c_3 e^{i\omega t + ikx}. \quad (5.87)$$

Note that the conservation equation $\omega \mathcal{P}_x + k\Pi_{xx} = 0$ is satisfied.

In summary, in the scalar sector, we have a three-parameter family of solutions of the equations of motion which satisfy our asymptotic boundary conditions. The stress tensor only depends on two of the parameters, and is finite and conserved, with all the components we would expect;

$$\mathcal{E} = kc_2' e^{i\omega t + ikx}, \qquad \mathcal{E}_x = -\omega c_2' e^{i\omega t + ikx}, \quad (5.88)$$

$$\mathcal{P}_x = kc_3' e^{i\omega t + ikx}, \qquad \Pi_{xx} = -\omega c_3' e^{i\omega t + ikx}, \qquad \Pi_{yy} = (zkc_2' + \omega c_3') e^{i\omega t + ikx}, \quad (5.89)$$

where to simplify the form of the stress tensor we write $c_2' = 2\frac{(z+2)}{z}kc_2$ and $c_3' = 2\frac{(z-1)(z+2)}{z}\omega k^2 c_3$.

5.2.2 Vector modes

Consider now the vector modes, described by the functions $v_1(r), v_2(r), v_3(r)$. The equations of motion for these are

$$\omega(r v_1' - r^{-2(z-1)} r v_2') = -k^2 r v_3', \quad (5.90)$$

$$r^2 v_1'' + (2z+1) r v_1' + z r^{-2(z-1)} r v_2' = \left(\frac{k^2}{r^2} - \frac{\omega^2}{r^{2z}} \right) v_1, \quad (5.91)$$

$$r^2 v_3'' + (z+3) r v_3' + \omega \frac{v_1}{r^2} - \omega \frac{v_2}{r^{2z}} = -\frac{\omega^2}{r^{2z}} v_3. \quad (5.92)$$

For the vector part, the non-zero parts of the stress tensor complex are

$$\mathcal{E}_y = r^{z+2} \left[r\partial_r v_2 + \frac{(z-2)}{z} r^{2(z-1)} r\partial_r v_1 \right] e^{i\omega t + ikx} + \mathcal{E}_y^{\text{deriv}}, \quad (5.93)$$

$$\mathcal{P}_y = r^{z+2} [-r\partial_r v_1 + r^{-2(z-1)} r\partial_r v_2] e^{i\omega t + ikx} + \mathcal{P}_y^{\text{deriv}}, \quad (5.94)$$

$$\Pi_{xy} = -r^{z+2} k r\partial_r v_3 e^{i\omega t + ikx} + \Pi_{xy}^{\text{deriv}}. \quad (5.95)$$

The first equation (5.90) imposes the conservation equation $\omega \mathcal{P}_y + k\Pi_{xy} = 0$.

Note that if $\omega = 0$, the first equation implies that $v_3' = 0$, and v_3 drops out of the system of equations — it vanishes up to a possible constant term. We will drop constant terms in v_1, v_2 and v_3 as not satisfying the asymptotic boundary conditions. Therefore v_3 will vanish if $\omega = 0$, so we rescale $v_3 \to \omega v_3$. Then the equations of motion are

$$rv_1' - r^{-2(z-1)} rv_2' = -k^2 rv_3', \tag{5.96}$$

$$r^2 v_1'' + (2z+1)rv_1' + zr^{-2(z-1)} rv_2' = \left(\frac{k^2}{r^2} - \frac{\omega^2}{r^{2z}}\right) v_1, \tag{5.97}$$

$$r^2 v_3'' + (z+3)rv_3' + \frac{v_1}{r^2} - \frac{v_2}{r^{2z}} = -\frac{\omega^2}{r^{2z}} v_3. \tag{5.98}$$

We again solve these equations perturbatively in k^2, ω^2, writing $F = \sum_{l,m} k^{2l}\omega^{2m} F^{(l,m)}$ and treating the r.h.s. as a source term for the solution at a given order determined in terms of the solution at earlier orders. At the leading order, the solution is

$$v_1^{(0,0)}(r) = \frac{c_4}{r^{3z}}, \quad v_2^{(0,0)}(r) = \frac{3z}{z+2}\frac{c_4}{r^{z+2}}, \quad v_3^{(0,0)}(r) = \frac{c_5}{r^{z+2}} + \frac{(z-1)}{z(z+2)(3z+2)}\frac{c_4}{r^{3z+2}}, \tag{5.99}$$

where we have once again set constant terms to zero by the boundary conditions. In terms of the stress tensor, the constant c_4 is associated with a finite contribution to $\mathcal{E}_y$, and c_5 is associated with a finite contribution to Π_{xy}. To evaluate the full stress tensor, we need some of the higher-order terms.

In this case, the equations are the same at each order, so there are no new homogeneous solutions; homogeneous solutions at higher order can be absorbed into a redefinition of c_4, c_5. We therefore need to consider only relevant particular integrals. The particular integrals from the solution parametrized by c_4 make no contribution to the stress tensor. However, the particular integrals $v_1^{(l,0)}$, $v_2^{(l,0)}$ for $l < z$ associated to the solution parametrized by c_5 will make potentially divergent contributions to $\mathcal{E}_y$. As in the constant perturbations, for $z \geq 4$, we will need to set $c_5 = 0$ to satisfy the boundary condition $v_2 \to 0$ as $z \to \infty$. The divergences then involve the particular integrals up to $l = 3$, which are

$$v_1^{(1,0)} = -\frac{c_5 z}{2(z-1)r^{z+2}}, \qquad v_2^{(1,0)} = -\frac{c_5(z^2-4)r^{z-4}}{2(z-4)(z-1)},$$
$$v_3^{(1,0)} = -\frac{c_5}{(z^2-16)r^{z+4}}, \tag{5.100}$$

$$v_1^{(2,0)} = \frac{3c_5 z}{4(z-1)(z^2-16)r^{z+4}}, \qquad v_2^{(2,0)} = \frac{c_5 r^{z-6}}{4(z-6)(z-1)}, \tag{5.101}$$

$$v_3^{(2,0)} = -\frac{c_5(z-8)}{8(z^2-16)(z^2-36)r^{z+6}}, \tag{5.102}$$

$$v_1^{(3,0)} = \frac{c_5 z(z-11)}{16(z-1)(z-3)(z+4)(z^2-36)r^{z+6}}, \qquad v_2^{(3,0)} = \frac{c_5(z^2-3z+8)r^{z-8}}{2(z-3)(z-8)(z^2-16)}, \tag{5.103}$$

$$v_3^{(3,0)} = \frac{c_5(5z^2-43z+72)}{24(z-3)(z^2-16)(z^2-36)(z^2-64)r^{z+8}}. \tag{5.104}$$

Note that at $z = 3$, this form for the particular integral will not apply, and it will be replaced by a solution involving logarithms, as occurred for $z = 2$ in the scalar sector. We have not determined this solution explicitly as this is not a particularly interesting value of z. We also need to consider the particular integral $v_i^{(0,1)}$ for the solution parametrized by c_5, as the contribution to v_2 would go like $r^{-(z+2)}$, and hence could make a finite contribution. A particular integral is

$$v_1^{(0,1)} = 0, \quad v_2^{(0,1)} = 0, \quad v_3^{(0,1)} = -\frac{c_5}{2z(3z+2)r^{3z+2}}, \tag{5.105}$$

so this will make no contribution to the stress tensor.

We can now use this to calculate the value of the contribution to the stress tensor from this mode for generic z. We have

$$\begin{aligned} \mathcal{E}_y &= r^{z+2}\left[r\partial_r v_2 + \frac{(z-2)}{z} r^{2(z-1)} r\partial_r v_1\right] e^{i\omega t+ikx} + \mathcal{E}_y^{\text{deriv}} \\ &= -\left[6(z-1)c_4 + \frac{c_5}{2(z-4)} r^{2z-4}k^4 - \frac{(z-5)c_5}{2(z-3)(z-6)(z^2-16)} r^{2z-6}k^6\right] e^{i\omega t+ikx} + \mathcal{E}_y^{\text{deriv}}, \end{aligned} \tag{5.106}$$

while from appendix A, we have

$$\begin{aligned} \mathcal{E}_y^{\text{deriv}} = c_5 \Bigg[&\left(-\frac{2\sigma_1}{(z-4)} - \sigma_2\right) k^4 r^{2z-4} \\ &+ \left(-\frac{(z-8)\sigma_1}{2(z-6)(z^2-16)} + \frac{3\sigma_2}{2(z^2-16)} - \sigma_3\right) k^6 r^{2z-6}\Bigg] e^{i\omega t+ikx}. \end{aligned} \tag{5.107}$$

The term at order k^4 grows at large r for $z > 2$, and the term at order k^6 grows at large r for $z > 3$. Since we can only consider this mode for $z < 4$, there are no further divergences. We can cancel the divergent terms by setting

$$\sigma_2 = -\frac{4\sigma_1 + 1}{2(z-4)}, \quad \sigma_3 = -\frac{(z-3)(7z-44)\sigma_1 + (z-1)(z-8)}{4(z-3)(z-4)(z-6)(z^2-16)}. \tag{5.108}$$

Thus, for generic z, we can obtain a finite stress tensor complex in the linearised approximation by choosing appropriate curvature counterterms in our definition of the action.[13] With this choice of action,

$$\mathcal{E}_y = -6(z-1)c_4 e^{i\omega t+ikx}. \tag{5.109}$$

For the other components, we have

$$\mathcal{P}_y = r^{z+2}[-r\partial_r v_1 + r^{-2(z-1)} r\partial_r v_2] e^{i\omega t+ikx} + \mathcal{P}_y^{\text{deriv}} = -(z+2)c_5 k^2 e^{i\omega t+ikx}, \tag{5.110}$$

and

$$\Pi_{xy} = -k\omega r^{z+2} r\partial_r v_3 e^{i\omega t+ikx} + \Pi_{xy}^{\text{deriv}} = (z+2)c_5 k\omega e^{i\omega t+ikx}. \tag{5.111}$$

We see in appendix A that the curvature components make no contributions to these components. As a consistency check, we see that $\omega\mathcal{P}_y + k\Pi_{xy} = 0$.

[13] For $z = 3$, a different choice of coefficients will be required.

In summary, for the non-zero momentum perturbations, we have a five-parameter family of solutions (parametrized by c_1, c_2 and c_3 in the scalar sector and c_4 and c_5 in the vector sector). For $z < 2$, in the scalar sector, we imposed a tighter boundary condition by setting $d_3 = 0$. For the non-zero momentum perturbations, this is required to get a finite energy density, and hence to satisfy $\delta S = 0$. Four of the parameters correspond to the independent components of the stress tensor in this non-zero momentum sector; as in the previous constant case, the scalar mode parametrized by c_1 does not contribute to the stress tensor complex at this linear order. For $z \geq 4$, we must again set $\mathcal{P}_i = 0$, and we are left with a three-parameter family of solutions.

Note that as we said earlier, for $z \geq 2$, this linearised calculation of the relation between the asymptotic behaviour of the metric and the action will not be reliable for a general perturbation. However, this calculation is always applicable if we consider a specific case where the bulk spacetime is everywhere a small perturbation away from the background. It is thus a significant result that can obtain a completely finite stress tensor complex at this linearised level by an appropriate choice of counterterms in the action.

5.3 Vanishing of the variation of the action

We return briefly to the question of the vanishing of the variation of the action. We have seen that all components of the stress tensor are finite. Thus, $\delta S = 0$ if $\delta\hat{e}^A_\alpha \to 0$ as $r \to \infty$. The coefficient of the variation of the vector field is

$$s_0 = -\alpha\left[zr^{z+2}\hat{a}_t + r^{z+2}r\partial_r\left(\frac{1}{2}\hat{h}_{tt} + \hat{a}_t\right) - r^2\partial_t\hat{a}_r\right] + s_0^{\text{deriv}}. \tag{5.112}$$

Since $\hat{h}_{tt}$ and $\hat{a}_t$ have components that go like $r^{-\frac{1}{2}(z+2+\beta_z)}$, s_0 will have a divergence like $r^{\frac{1}{2}(z+2-\beta_z)}$, which gives a positive power of r for $1 \leq z < 2$. (For $z = 2$, this is replaced by a $\ln r$ divergence). However, it is precisely for this range that we impose the stronger boundary condition, which implies that δA^0 vanishes more quickly than $r^{-\frac{1}{2}(z+2-\beta_z)}$ as $r \to \infty$. This is precisely what is required to ensure that the $s_0\delta A^0$ contribution also vanishes, so we indeed satisfy $\delta S = 0$ on-shell.

Thus, for our asymptotic boundary conditions, (2.7) is a good action principle for the asymptotically Lifshitz spacetimes, as it is finite on-shell and satisfies $\delta S = 0$ for arbitrary variations satisfying the boundary conditions.

5.4 Operator dual to A^0

We have shown that the stress tensor is finite for our action. We should also consider the operator dual to A^0, and see if its expectation value is finite. As we remarked earlier, for $1 \leq z < 2$, it seems natural to think of the part of A^0 falling off as $r^{-\frac{1}{2}(z+2-\beta_z)}$ as a non-normalizable mode (that is, as boundary data associated with the vector field). If we write $\delta A^0 = r^{-\frac{1}{2}(z+2-\beta_z)}\delta\bar{A}^0$, $\bar{s}_0 = r^{\frac{1}{2}(z+2-\beta_z)}s_0$ is the coefficient of $\delta\bar{A}^0$ in the variation of the action, which would be interpreted as the expectation value of the dual operator. The term falling off as $r^{-\frac{1}{2}(z+2+\beta_z)}$ makes a finite contribution to $\bar{s}_0$, so it can be thought of as the corresponding normalizable mode. In our linearised analysis, this implies that the additional scalar mode which does not contribute to the stress tensor can be interpreted as the expectation value of the operator dual to changes in the non-normalisable mode for the vector field.

However, it is difficult to extend this analysis to $z \geq 2$. The mode which falls off like $r^{-\frac{1}{2}(z+2-\beta_z)}$ then violates our boundary conditions for the metric components, so it is not clear if it can still be interpreted as boundary data for the vector field. If we calculate $\bar{s}_0$ anyway, it has a finite contribution from the mode which falls off as $r^{-\frac{1}{2}(z+2+\beta_z)}$, which suggests this mode can be given the same interpretation, but it now also has a divergent contribution from the mode which falls off as $r^{-(z+2)}$. If we want to think of the mode in our linearised analysis which falls off as $r^{-\frac{1}{2}(z+2-\beta_z)}$ as the boundary data we are varying, then because this mode appears in f and k as well as j, the coefficient of this variation is really a linear combination of $\bar{s}_0$, $\mathcal{E}$ and Π_i^i (this didn't make any difference for $1 \leq z < 2$ because the contribution from the mode which falls off as $r^{-(z+2)}$ vanished). However, this does not seem to cancel the divergence. There is a combination of $\bar{s}_0$, $\mathcal{E}$ and Π_i^i which will cancel the divergent contribution from the mode which falls off as $r^{-(z+2)}$, but the coefficients are different from those implied by the solution of the linearised equations. We leave the resolution of this conundrum for future work.

6 Discussion

The main results of this paper are that first, we have constructed an appropriate action principle for asymptotically Lifshitz spacetimes with a flat boundary in the massive vector theory of [31]. We then proposed a definition of the non-relativistic stress tensor complex for both the Schrödinger and Lifshitz cases in terms of the variation of the action. Our proposal corresponds to the proposal of [30] in the relativistic case, taking the appropriate variation to be a variation of the boundary frame fields holding the matter fields with tangent space indices fixed. This is one of our key results: the major difference between the calculation of the stress tensor in these cases and the more familiar AdS case is not the different scaling of different directions, but simply the fact that we need to take the contribution to the stress tensor from variation of the vector field into account. Once we have correctly accounted for this, we get finite answers for the stress tensor complex.

In the Schrödinger case, we have shown that this proposal agrees with the stress tensor complex obtained by re-interpreting the stress tensor of the related asymptotically AdS spacetime in terms of the non-relativistic field theory [23], for asymptotically Schrödinger spacetimes which can be obtained by TsT transformation from a vacuum asymptotically AdS spacetime. We expect this will be true in general, but have left the detailed calculation for future work. In the Lifshitz case, we have solved the linearised equations of motion for the general perturbation about the background (2.4). This enables us to relate the stress tensor to the asymptotic falloff of the metric and vector fields of an asymptotically Lifshitz spacetime. We have shown that the resulting stress tensor is finite.

There are a number of interesting directions for future work. For the Schrödinger case, it would be useful to establish the minimal boundary conditions for which we have a well-defined action principle, parallelling our analysis for the Lifshitz case. Our results on finiteness of the stress tensor imply that we can relax the boundary conditions somewhat relative to those used in [25], but as in the Lifshitz case, there are divergences in the matter sector that need to be addressed. There has been extensive work on obtaining Schrödinger geometries in different contexts [38–42], and it would be useful to work out the boundary counterterms required to construct appropriate action principles in these different cases. The fact that we now have a proposal for constructing the stress tensor directly in the asymptotically Schrödinger solution is particularly useful for these cases where we do not have a solution-generating transformation relating asymptotically AdS and asymptotically Schrödinger solutions.

For the asymptotically Lifshitz geometries, to complete the analysis of one-point functions, we need to resolve the problems with the calculation of the expectation value for the operator dual to the vector field raised in the previous subsection. It would also be interesting to use our action to calculate two-point functions in the pure Lifshitz background. In the study of finite-temperature geometries, it would be interesting to further pursue the holographic renormalization framework by understanding the construction of more general black hole solutions corresponding to arbitrary hydrodynamic stress tensors from the dual field theory point of view. A natural next step is to construct a black hole with non-zero spatial momentum. Because the background geometry does not have a boost invariance, this cannot be obtained by simply boosting the known solutions.

Lifshitz points usually occur at the juncture of three phase boundaries. There is more than one ordered phase below a critical temperature. Depending on an external control parameter, for instance an external field, one can make a transition from an ordered phase where the condensate is spatially uniform to a new ordered phase where the order is inhomogeneous; that is, the critical system is allowed to have a phase where the Landau energy of the system is minimized by a non-uniform condensate rather than a homogeneous one [43]. Lifshitz critical points are relevant for studying interesting condensed matter systems including superconductors and Liquid Crystals among others [44] and [45]. Recently hairy black holes in AdS have been of central importance in modeling second order transitions in the context of AdS/CFT (notably superfluid/superconductor transition). The second order transition was modeled by a charged scalar condensing in the vicinity of a black hole event horizon [46, 47] in AdS. Perhaps the first step to model a Lifshitz point at finite temperature is to study a similar set up but with the bulk black hole replaced with a Lifshitz black hole; one should see if a hairy Lifshitz black hole with $z \neq 1$ can be constructed.

For both cases, it would be interesting to extend the analysis to consider more general boundary data. We have restricted ourselves to the case where the boundary is flat, but a similar definition of the stress tensor can be applied for a general curved spatial metric g_{ij}. The most interesting case to consider is when the boundary metric is a sphere. This introduces additional slow falloff terms in the asymptotics, so we would need to check again that the resulting stress tensor is finite, and hence that we have a well-defined action principle for such boundary conditions. The perturbation analysis for asymptotically Lifshitz spacetimes with a spherical boundary has been initiated in [27, 29].

At a more formal level, we would like to have a better understanding of the possible boundary data for asymptotically Lifshitz spacetimes. In particular, there are issues we have not yet fully understood about the meaning of our calculation of the energy flux

JHEP09(2009)009

from the point of view of a non-relativistic theory. By analogy with the relativistic case, we have constructed our stress tensor by considering arbitrary variations of the boundary data, including variations $\delta\hat{e}_i^{(0)}$, which give the energy flux. However, introducing such components does not seem natural from the point of view of a non-relativistic theory. In a non-relativistic theory, the flat background spacetime we considered above can be thought of as a fiber bundle, with the spatial slices fibered over the time direction [48]. Each spatial slice corresponds to a moment in time, and relative position in the spatial slices is invariantly defined, but there is no invariant notion of relative position in different spatial slices. Allowing components $\hat{e}_t^{(i)}$ is consistent with this fiber bundle structure, but it is not clear how $\hat{e}_i^{(0)}$ would be. It would be interesting to understand this distinction between the different components of the stress tensor complex more fully.

Another general issue is to find a truncation of string theory which gives a Lifshitz geometry with anisotropic scaling symmetry. That is, where the metric takes the form (2.1), and the matter fields are also invariant under the isometry $t \to \lambda^z t$, $x^i \to \lambda x^i$, $r \to \lambda^{-1} r$. The symmetry implies that any scalar field must be a constant, which makes it difficult to find an embedding in string theory, where a timelike vector field is usually accompanied by a non-trivial scalar.

An important general issue for applications of holography to condensed matter systems is that it is not generally understood what the conditions are under which the theory has a classical weakly curved gravitational dual. That is, what is the analogue of the large N limit for gauge theories which implies that quantum corrections to the gravity theory under control?

Acknowledgments

We are grateful for useful discussions with Simon Caron-Huot, Veronika Hubeny, Alex Maloney, Don Marolf and Mukund Rangamani, and thank Don Marolf and Mukund Rangamani for reading a draft of this paper. SFR is supported by STFC. OS is supported by NSERC of Canada.

A Derivative contributions to the Lifshitz stress tensor

In this appendix, we will evaluate the contributions of the part of the boundary action involving derivatives to the stress tensor for the asymptotically Lifshitz spacetimes in our linearised perturbative analysis. We will not discuss the most general possible derivative terms, but consider a simple set of terms up to fourth order in derivatives which are sufficient to cancel the divergences in $\mathcal{E}_y$, giving us a finite stress tensor complex for the linearised perturbations. The form of the action will not be uniquely fixed by imposing finiteness of the stress tensor; our aim here is simply to show that there is a choice for the counterterms S_{deriv} which gives a finite answer for the stress tensor. We consider an action

$$S_{\text{deriv}} = \frac{1}{16\pi G_4} \int d^3\xi \sqrt{-h}[\sigma_1 R^h + \sigma_2 \nabla_\alpha A_\beta \nabla^\alpha A^\beta + \sigma_3 (\Box A_\alpha)(\Box A^\alpha)], \tag{A.1}$$

where R^h is the curvature of the boundary metric, and the σ_i are arbitrary constants.

We get no contribution from $S_{\rm deriv}$ for constant perturbations. For the non-zero momentum modes considered in section 5.2, the contribution to the boundary stress tensor complex becomes

$$\begin{aligned}
\mathcal{E}^{\rm deriv} &= -r^z k^2[\sigma_1(k_L - k^2 k_T) + 2\alpha^2\omega(\sigma_2 - \sigma_3\Box)s_1]e^{i\omega t+ikx}, \\
\mathcal{E}_x^{\rm deriv} &= r^z k\omega[\sigma_1(k_L - k^2 k_T) + 2\alpha^2\omega(\sigma_2 - \sigma_3\Box)s_1]e^{i\omega t+ikx}, \\
\mathcal{E}_y^{\rm deriv} &= -\left[\sigma_1(r^z(k^2\omega v_3 - k^2 v_2) + r^{3z-2}k^2 v_1) - \alpha^2 r^{3z}\left(\frac{k^2}{r^2} - 2\frac{\omega^2}{r^{2z}}\right)(\sigma_2 - \sigma_3\Box)v_1\right]e^{i\omega t+ikx}, \\
\mathcal{P}_x^{\rm deriv} &= r^z k[\sigma_1 r^{-2z+2}\omega(k_L - k^2 k_T) + 2\alpha^2 k^2(\sigma_2 - \sigma_3\Box)s_1]e^{i\omega t+ikx}, \\
\mathcal{P}_y^{\rm deriv} &= -r^z k^2[\sigma_1(r^{-2z+2}(\omega v_3 - v_2) + v_1) + \alpha^2(\sigma_2 - \sigma_3\Box)v_1]e^{i\omega t+ikx}, \\
\Pi_{xx}^{\rm deriv} &= -r^z\omega[\sigma_1 r^{-2z+2}\omega(k_L - k^2 k_T) + 2\alpha^2 k^2(\sigma_2 - \sigma_3\Box)s_1]e^{i\omega t+ikx}, \\
\Pi_{xy}^{\rm deriv} &= r^z k\omega[\sigma_1(r^{-2z+2}(\omega v_3 - v_2) + v_1) + \alpha^2(\sigma_2 - \sigma_3\Box)v_1]e^{i\omega t+ikx}, \\
\Pi_{yy}^{\rm deriv} &= \sigma_1[r^z(-k^2 f - 2k^2\omega s_1) + r^{2-z}(2k^2\omega s_2 - \omega^2 k_L - k^2\omega^2 k_T)]e^{i\omega t+ikx},
\end{aligned} \tag{A.2}$$

where for the plane wave perturbations, $\Box = \frac{\omega^2}{r^{2z}} - \frac{k^2}{r^2}$. We can see immediately that this contribution to the stress tensor is separately conserved, as we would expect: $\omega\mathcal{E}^{\rm deriv} + k\mathcal{E}_x^{\rm deriv} = 0$, $\omega\mathcal{P}_x^{\rm deriv} + k\Pi_{xx}^{\rm deriv} = 0$, $\omega\mathcal{P}_y^{\rm deriv} + k\Pi_{xy}^{\rm deriv} = 0$. Note that this did not require the use of the equations of motion, unlike for the part of the action we treated in the body of the paper.

The contributions to most components of the stress tensor complex from the derivative terms will vanish. The general point is that the derivative terms are suppressed relative to the terms considered earlier by factors of k^2/r^2 or ω^2/r^{2z}. Hence when the earlier terms give finite contributions, the derivative terms will give vanishing contributions. Explicitly, the scalar components $\mathcal{E}^{\rm deriv}, \mathcal{E}_x^{\rm deriv}, \mathcal{P}_x^{\rm deriv}, \Pi_{xx}^{\rm deriv}$, and $\Pi_{yy}^{\rm deriv}$ involve $r^z f$, $r^z k_L$, $r^z k_T$, $r^z s_1$ and $r^z s_2$, (or smaller powers of r) all of which vanish for the general solution of the linearised equations satisfying our boundary conditions obtained in section 5.2. Similarly, for the vector sector, the components $\mathcal{P}_y^{\rm deriv}$ and $\Pi_{xy}^{\rm deriv}$ involve $r^{-z+2}v_3$, $r^{-z+2}v_2$ and $r^z v_1$, all of which vanish for the general solution of the linearised equations satisfying our boundary conditions obtained in section 5.2.

The one exception is $\mathcal{E}_y$, which involves $r^z v_3$, $r^z v_2$, and $r^{3z-2}v_1$, which vanish for the solution parametrized by c_4, but not for that parametrized by c_5. This is precisely where we found divergences for the terms coming from the non-derivative part of the action, so we want to evaluate the derivative terms and see that we can choose the coefficients to cancel these divergences. It is the v_1 and v_2 terms which produce potential divergences; there is a term which goes like $k^4 r^{2z-4}$ from putting $v_i^{(1,0)}$ in the σ_1, σ_2 terms, and terms that go like $k^6 r^{2z-6}$ from putting $v_i^{(2,0)}$ in the σ_1, σ_2 terms and from putting $v_i^{(1,0)}$ in the σ_3 term. As we only have a c_5 mode for $z < 4$, these are the only potential finite or divergent terms. Putting them together, the divergent terms for this mode are

$$\begin{aligned}
\mathcal{E}_y^{\rm deriv} = c_5 \Bigg[&\left(-\frac{2\sigma_1}{(z-4)} - \sigma_2\right) k^4 r^{2z-4} \\
&+ \left(-\frac{(z-8)\sigma_1}{2(z-6)(z^2-16)} + \frac{3\sigma_2}{2(z^2-16)} - \sigma_3\right) k^6 r^{2z-6}\Bigg] e^{i\omega t+ikx}.
\end{aligned} \tag{A.3}$$

The terms we are omitting in this expression vanish as $r \to \infty$ for $z < 4$. We can then choose the coefficients σ_i to cancel these divergences against the divergences in $\mathcal{E}_y$ from the non-derivative part of the action; we do this explicitly in section 5.2. Since there are two divergences to cancel and three coefficients, this will not fix the form of the action uniquely. Thus, there is an action for which the stress tensor is finite, but this condition does not determine a unique choice of the action. We kept two terms at second order in derivatives in our discussion of $S_{\rm deriv}$ to illustrate this failure to fix a unique action.

If we considered more general boundary data, such as where the spatial metric is replaced by a sphere, there will be further constraints on the coefficients in the derivative terms. For example, if we consider a Lifshitz spacetime where the spatial sections are spheres, the term involving the curvature of the boundary metric will contribute to the action, but the terms involving derivatives of the vector field will not. The coefficient of the curvature term can then be fixed by cancelling the divergence in the on-shell action arising from the new terms in the metric at relative order $1/r^2$. We leave a detailed discussion of the extension of our analysis to more general boundary data for future work.

B Euclidean action and thermodynamic energy

In this section, we show that our definition of the energy density for asymptotically Lifshitz spacetimes agrees with the thermodynamic energy density obtained by using the Euclidean version of the black hole solution as a saddle-point in the path integral for a class of static asymptotically Lifshitz black hole spacetimes. Our analysis in this section will not use the linearised analysis we used previously; we find that we can rewrite the action for the black hole solutions we consider in an appropriate form just by using the equations of motion (2.2), (2.3).

We consider a metric ansatz

$$ds^2 = -p(r)dt^2 + q(r)(dx^2 + dy^2) + \frac{dr^2}{r^2}, \quad A_t = A_t(r). \tag{B.1}$$

This will give a black hole solution if there is an event horizon at $r = r_H$, where $p(r) = p_H(r - r_H)^2 + \mathcal{O}(r - r_H)^3$. For a regular horizon, we must also have $A_t(r) = A_{tH}(r - r_H) + \mathcal{O}(r - r_H)^3$. We assume that there is a solution of the equations of motion with these properties; such solutions were constructed numerically in [27, 28].

If we rotate $t \to -i\tau$, the Euclidean black hole solution gives a saddle-point approximation to the path integral defining the thermal partition function at temperature

$$T_H = r_H \frac{\sqrt{p_H}}{2\pi}. \tag{B.2}$$

In the Euclidean black hole solution, the radial coordinate is restricted to $r_H \leq r < \infty$, with a smooth origin at $r = r_H$ once we choose $\Delta\tau = \beta = T_H^{-1}$. The action for this black hole solution gives an approximation to the free energy, $F = T_H I_{\rm Eucl}$. Since the solution is translationally invariant in x and y, it is natural to divide by the coordinate volume in those directions to define the free energy density $f = T_H I_{\rm Eucl}/V_2$. The entropy density is given by the area of the black hole horizon,

$$s = \frac{A}{4G_4} = \frac{q(r_H)}{4G_4}, \tag{B.3}$$

so we can define the thermodynamic energy density by

$$f = \mathcal{E}_{\rm thermo} - T_H s. \tag{B.4}$$

We want to see that this agrees with the energy density we defined previously, $\mathcal{E}_{\rm thermo} = \mathcal{E}$.

To do so, we use the equations of motion to rewrite the on-shell Euclidean action (2.7) in terms of boundary terms at the asymptotic boundary and at the horizon.

After the analytic continuation $t \to -i\tau$, the action of the Euclidean solution is

$$\begin{aligned} I^E =& -\frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right) \\ & -\frac{1}{16\pi G_4}\int d^3\xi\sqrt{h}(2K - 4 - z\alpha\sqrt{-A_\alpha A^\alpha}) - I_{\rm deriv}. \end{aligned} \tag{B.5}$$

To relate the action to the boundary terms, it is convenient to use the equation of motion for the vector field (2.3) to write $m^2 A_\mu A^\mu + \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \nabla_\mu(F^\mu_\nu A^\nu)$, so

$$\begin{aligned} I^E =& -\frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left(R - 2\Lambda + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu\right) \\ & +\frac{1}{16\pi G_4}\int d^3\xi\sqrt{h}(n^\mu F_{\mu\nu}A^\nu - 2K + 4 + z\alpha\sqrt{-A_\alpha A^\alpha}) - I_{\rm deriv}. \end{aligned} \tag{B.6}$$

Now for the ansatz (B.1), the derivative terms do not contribute and the only non-zero component of $F_{\mu\nu}$ is $F_{rt} = -F_{tr}$, so $A_t A^t = A_\mu A^\mu$ and $F_{rt}F^{rt} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, and hence the Einstein equations (2.2) imply

$$R^x{}_x + R^y{}_y + R^r{}_r - R^t{}_t = 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{B.7}$$

Thus, the on-shell action for the ansatz (B.1) is

$$I^E = -\frac{1}{16\pi G_4}\int d^4x\sqrt{g}2R^t_t + \frac{1}{16\pi G_4}\int d^3\xi\sqrt{h}(n^\mu F_{\mu\nu}A^\nu - 2K + 4 + z\alpha\sqrt{-A_\alpha A^\alpha}). \tag{B.8}$$

Furthermore, using the form of the metric (B.1), we can show

$$\sqrt{g}R^t{}_t = -(\sqrt{h}K^t{}_t)', \tag{B.9}$$

so the integration over r can be rewritten in terms of boundary terms. The integration over τ and x, y gives an overall factor of βV_2, which we divide out. Thus,

$$16\pi G_4\frac{I^E}{\beta V_2} = \sqrt{h}(2K^t{}_t + n^\mu F_{\mu\nu}A^\nu - 2K + 4 + z\alpha\sqrt{-A_\mu A^\mu})\,|_{r_b} - 2\sqrt{h}K^t{}_t\,|_{r_H}, \tag{B.10}$$

where r_b and r_H are the location of the boundary and the horizon respectively. At the horizon,

$$2\sqrt{h}K^t{}_t|_{r_H} = r\frac{q(r)p(r)'}{\sqrt{p(r)}}|_{r_H} = 2r_H\sqrt{p_H}q(r_H) = 16\pi G_4 T_H s, \tag{B.11}$$

so the surface term at the horizon reproduces the term $-T_H s$ in the free energy density. The surface term at infinity is hence giving the thermodynamic energy density. Now using (2.8), (2.9),

$$\mathcal{E} = 2s^t{}_t - s^t A_t = \sqrt{-h}(2K^t{}_t - 2K + 4 + n^\mu F_\mu{}^\nu A_\nu + z\alpha\sqrt{-A_\mu A^\mu})|_{r_b}, \tag{B.12}$$

so the surface term at infinity in (B.10) is precisely our energy density; that is, $\mathcal{E}_{\text{thermo}} = \mathcal{E}$ as desired.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].

[2] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [SPIRES].

[3] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [hep-th/0209067] [SPIRES].

[4] S.S. Gubser and A. Karch, *From gauge-string duality to strong interactions: a Pedestrian's Guide*, arXiv:0901.0935 [SPIRES].

[5] M. Rangamani, *Gravity & hydrodynamics: lectures on the fluid-gravity correspondence*, arXiv:0905.4352 [SPIRES].

[6] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Building a holographic superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601 [arXiv:0803.3295] [SPIRES].

[7] C.P. Herzog, P.K. Kovtun and D.T. Son, *Holographic model of superfluidity*, arXiv:0809.4870 [SPIRES].

[8] P. Kovtun and D. Nickel, *Black holes and non-relativistic quantum systems*, *Phys. Rev. Lett.* **102** (2009) 011602 [arXiv:0809.2020] [SPIRES].

[9] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, arXiv:0903.3246 [SPIRES].

[10] C.R. Hagen, *Scale and conformal transformations in galilean-covariant field theory*, *Phys. Rev.* **D 5** (1972) 377 [SPIRES].

[11] T. Mehen, I.W. Stewart and M.B. Wise, *Conformal invariance for non-relativistic field theory*, *Phys. Lett.* **B 474** (2000) 145 [hep-th/9910025] [SPIRES].

[12] Y. Nishida and D.T. Son, *Nonrelativistic conformal field theories*, *Phys. Rev.* **D 76** (2007) 086004 [arXiv:0706.3746] [SPIRES].

[13] R.M. Hornreich, M. Luban and S. Shtrikman, *Critical behavior at the onset of $\vec{k}$-space instability on the λ line*, *Phys. Rev. Lett.* **35** (1975) 1678 [SPIRES].

[14] G. Grinstein, *Anisotropic sine-Gordon model and infinite-order phase transitions in three dimensions*, *Phys. Rev.* **B 23** (1981) 4615.

[15] C.L. Henley, *Relaxation time for a dimer covering with height representation*, *J. Stat. Phys.* **89** (1997) 483.

[16] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [arXiv:0804.3972] [SPIRES].

[17] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [arXiv:0804.4053] [SPIRES].

[18] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev.* **D 78** (2008) 106005 [arXiv:0808.1725] [SPIRES].

[19] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087] [SPIRES].

[20] V. Balasubramanian and P. Kraus, *A stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [hep-th/9902121] [SPIRES].

[21] S. Bhattacharyya, V.E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear fluid dynamics from gravity*, *JHEP* **02** (2008) 045 [arXiv:0712.2456] [SPIRES].

[22] S. Bhattacharyya et al., *Local fluid dynamical entropy from gravity*, *JHEP* **06** (2008) 055 [arXiv:0803.2526] [SPIRES].

[23] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non- relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100] [SPIRES].

[24] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111] [SPIRES].

[25] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099] [SPIRES].

[26] M. Rangamani, S.F. Ross, D.T. Son and E.G. Thompson, *Conformal non-relativistic hydrodynamics from gravity*, *JHEP* **01** (2009) 075 [arXiv:0811.2049] [SPIRES].

[27] U.H. Danielsson and L. Thorlacius, *Black holes in asymptotically Lifshitz spacetime*, *JHEP* **03** (2009) 070 [arXiv:0812.5088] [SPIRES].

[28] R.B. Mann, *Lifshitz topological black holes*, *JHEP* **06** (2009) 075 [arXiv:0905.1136] [SPIRES].

[29] G. Bertoldi, B.A. Burrington and A. Peet, *Black holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent*, arXiv:0905.3183 [SPIRES].

[30] S. Hollands, A. Ishibashi and D. Marolf, *Counter-term charges generate bulk symmetries*, *Phys. Rev.* **D 72** (2005) 104025 [hep-th/0503105] [SPIRES].

[31] M. Taylor, *Non-relativistic holography*, arXiv:0812.0530 [SPIRES].

[32] R.B. Mann and D. Marolf, *Holographic renormalization of asymptotically flat spacetimes*, *Class. Quant. Grav.* **23** (2006) 2927 [hep-th/0511096] [SPIRES].

[33] L.D. Landau and E. M. Lifshitz, *Fluid Mechanics*, volume 6 of *Course of Theoretical Physics*, Pergamon Press, London U.K. (1959).

[34] A. Adams, A. Maloney, A. Sinha and S.E. Vazquez, $1/N$ *effects in non-relativistic gauge-gravity duality*, *JHEP* **03** (2009) 097 [arXiv:0812.0166] [SPIRES].

[35] S. Schäfer-Nameki, M. Yamazaki and K. Yoshida, *Coset construction for duals of non-relativistic CFTs*, arXiv:0903.4245 [SPIRES].

[36] D. Yamada, *Thermodynamics of black holes in Schrödinger space*, *Class. Quant. Grav.* **26** (2009) 075006 [arXiv:0809.4928] [SPIRES].

JHEP09(2009)009

[37] T. Azeyanagi, W. Li and T. Takayanagi, *On string theory duals of Lifshitz-like fixed points*, *JHEP* **06** (2009) 084 [arXiv:0905.0688] [SPIRES].

[38] P. Kovtun and D. Nickel, *Black holes and non-relativistic quantum systems*, *Phys. Rev. Lett.* **102** (2009) 011602 [arXiv:0809.2020] [SPIRES].

[39] S.A. Hartnoll and K. Yoshida, *Families of IIB duals for nonrelativistic CFTs*, *JHEP* **12** (2008) 071 [arXiv:0810.0298] [SPIRES].

[40] E.O. Colgain and H. Yavartanoo, *NR CFT_3 duals in M-theory*, arXiv:0904.0588 [SPIRES].

[41] N. Bobev, A. Kundu and K. Pilch, *Supersymmetric IIB solutions with Schrödinger Symmetry*, *JHEP* **07** (2009) 107 [arXiv:0905.0673] [SPIRES].

[42] A. Donos and J.P. Gauntlett, *Solutions of type IIB and $D = 11$ supergravity with Schrödinger(z) symmetry*, *JHEP* **07** (2009) 042 [arXiv:0905.1098] [SPIRES].

[43] R.M. Hornreich, M. Luban and S. Shtrikman, *Critical behavior at the onset of $\vec{k}$-space instability on the λ line*, *Phys. Rev. Lett.* **35** (1975) 1678 [SPIRES].

[44] N.G.B. Schuh, *Lifshitz-point behaviour of ferromagnetic superconductors*, *Z. Phys. B Con. Mat.* **B 46** (1982) 149.

[45] I. Muševič et al., *Phase diagram of a ferroelectric chiral smectic liquid crystal near the Lifshitz point*, *Phys. Rev. Lett.* **48** (1982) 192.

[46] S.S. Gubser, *Breaking an Abelian gauge symmetry near a black hole horizon*, *Phys. Rev.* **D 78** (2008) 065034 [arXiv:0801.2977] [SPIRES].

[47] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Building a holographic superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601 [arXiv:0803.3295] [SPIRES].

[48] R. Penrose, *The road to reality*, Jonathan Cape, London U.K. (2004).